Abstract: In this paper, we look for long run and short run effects of fiscal deficits on economic growth. We extend the Barro (1990) endogenous growth model with productive public spending to public deficit and debt. The model shows a multiplicity of long run balanced growth paths (a high-growth and a low-growth steady states), and a possible indeterminacy of the transition path, which may be consistent with empirical literature that exhibits strong non-linear responses of economic growth to fiscal deficits. Under very general hypothesis, our model shows that permanent deficits lead to a lower balanced growth path in the long run. In the short run, on the other hand, according to multiplicity, the effect of public deficit impulses depends on the initial level of public debt. Starting from the high-growth steady state, the rate of economic growth may initially increase, but, starting from the low-growth steady state, the effect of a fiscal deficit impulse is subjected to expectations on public debt sustainability.

JEL classification: H62, H63, E62

Keywords: deficit, productive government spending, endogenous growth, non-linear effects of fiscal policy, public debt

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Since the mid-seventies, OECD countries have been characterized by persistent deficits (see Table 1). Such deficits over a long period have probably affected the balanced growth path of these countries, but there is no unanimous theoretical or empirical answer to this question.

Table 1 – Public deficit (percentage of GDP) in OECD countries (1971 – 2008)

Concerning the long run effect of fiscal deficits, two points of view emerge. The first postulates that a higher deficit today means a higher debt interest burden tomorrow, with a crowding out effect on resources and thus on the balanced growth path. The second argues that a higher deficit may provide resources for productive public expenditures, which may improve the balanced growth path in the long run. In discussing the fiscal rules set in the Treaty of Maastricht and in the Stability and Growth Pact (hereafter SGP), for example, some authors have pointed out the risk that these rules may permanently reduce public investment and the rate of economic growth, and have suggested that public debt may be used as a way of finance for public infrastructures.\(^1\)

As regards the short run perspective, a number of recent empirical and theoretical papers try to identify so-called “Neoclassical”, “Ricardian” or “Neokeynesian” effects of fiscal deficit on economic growth.\(^2\) These studies, initiated by Feldstein (1982), Giavazzi & Pagano (1990) and Blanchard (1990), indicate that fiscal deficits seem associated with strong non-linear effects on growth, probably in line with expectations switches and initial debt level. Thus, fiscal impulses may have traditional “Keynesian” effects or reversed effects, depending on government financial stance. For example, in high-debt contexts, a fiscal

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\(^1\) Since achieving the deficit target by a cut in public investment may be easier than by a cut in “unproductive” public expenditures (like wages and transfer payments), but to the detriment of economic growth (Alesina & Perotti, 1997), public debt might be issued to finance productive expenditures. This idea is known as the “Golden Rule of Public Finance”, see e.g. Modigliani et al. (1998), Balassone & Franco (2000), Buiter (2001) and Blanchard & Giavazzi (2004). See also Minea & Villieu (2009a) for a critical view on the GRPF.

\(^2\) see, e.g. Perotti (1999), Giavazzi, Jappelli & Pagano (2000) or Adam & Bevan (2005), and the survey of Elmendorf & Mankiw (1999).
correction may reduce the probability of public sector default, thus improving confidence and increasing consumption and investment (Perotti, 1999).³

This paper attempts to join these two streams of literature (short run and long run) about the effects of fiscal deficits on economic growth. To study the long run implications of persistent deficits, we must allow for growth in the long run, because considering persistent deficits implies a perpetual public debt growth which will lead to an explosion of the public debt to output ratio in the long-run if steady-state output is constant. Second, issuing debt is costly, since government must pay interest payments, thus one must investigate the allocation of these resources. In order to give public debt a chance to increase economic growth, we consider that deficits are devoted to productive public expenditures (or public investment). A simple way to fulfill these objectives is to consider the Barro (1990) endogenous growth model with productive public spending. In Barro (1990), neither public debt, nor public deficits are allowed, thus all public expenditures are productive and growth-enhancing.⁴ To come close to data, we introduce persistent deficits, one immediate consequence being the appearance of unproductive public spending, in the form of interest payments on public debt. By so doing, the model is able to deal with the trade-off between the resources provided by fiscal deficits and the cost of the debt burden in the long run. Moreover, introducing public debt into Barro model generates multiplicity of balanced long-run growth paths, thus non-linearities of fiscal policy.

Compared with the large number of empirical studies, there are, to our knowledge, only a few papers that deal with persistent deficits in growth models. Non-linear effects of fiscal policy are studied by Sutherland (1997) in a stochastic continuous time model, adding to the previous work of Bertola & Drazen (1993), but in a no-growth context. In an interesting paper, Park & Philippopoulos (2004) address the question of indeterminacy of fiscal policy in an endogenous growth model, but their model presents no large interest in the deficit-to-growth relation. In a growth model close to our approach, Greiner & Semmler (2000), and Minea & Villieu (2009a) have a close look to the effect of public deficit increases devoted to public investment, but without reference to non-linearity or indeterminacy.

In our paper, as in Barro (1990), endogenous growth is achieved by introducing public productive spending in a constant-return-to-scale production function. In contrast with Barro

³ Although Giavazzi, Jappelli & Pagano (2000) do not find any link between the initial level of public debt and non-linear effects of fiscal policy, in contrast with Perotti (1999).
⁴ In Barro (1990), productive expenditures have a flow dimension and are not strictly speaking of public investment nature. However, our model can be easily extended to stock productive expenditures (public investment), as we shall see.
(1990), we allow for debt-financed public spending in the government budget constraint, thus providing a second dynamic relation, besides the Keynes-Ramsey rule. Persistent deficits are introduced via a “deficit rule”, namely a $m$ deficit-to-output ratio (to illustrate, in Table 1, $m = 2.5\%$). Under such an assumption, the productive public expenditures’ path will be endogenously determined in the government budget constraint.$^5$

Our results are twofold. First, as regards the steady state balanced growth path, we show that a “Barro-type” balanced budget rule ($m=0$) generates two long-run steady state equilibria: the “Barro solution”, with zero public debt, and a “Solow solution”, with zero growth. Allowing for deficits ($m>0$) yields two endogenous long-run positive growth paths, a high-growth path and a low-growth path. We show in particular that, similarly to Barro (1990), there is an inverted-U relation between the tax rate and the high-growth path, but the tax rate that maximizes economic growth is higher than Barro’s one, namely the elasticity of output to public expenditures in the production function. We find that in the long run, raising deficits (all things being equal) is always long run growth-reducing. The reason is straightforward: if Ponzi-finance is forbidden (governments are not allowed to finance the interest payment on debt by issuing new debt – deficit), the interest burden is higher than deficit resources in the long run, and government must adjust (diminish) productive expenditures in the long run, in order to respect the no-Ponzi finance constraint. To see things differently, initially devoted to growth-enhancing public deficits generate less resources than their cost in the long-run, for no-Ponzi finance to hold. With other things being equal, the government must shrink productive spending below their initial level, in order to finance the gap between deficit resources and their cost (interest burden) in the long run. Therefore, deficit rules always crowd out productive expenditures in the steady state.

Second, as regards transitional dynamics, the multiplicity of steady states may generate non-linear effects of fiscal impulses. We show that the low steady state is unstable and that the high steady state is a saddle point. Starting from the high-growth steady state, deficit rules may enhance economic growth in the short and medium run. Starting from the low-growth steady state, indeterminacy occurs: the economic growth may positively or negatively respond to fiscal deficit impulses. Consequently, the effect of a fiscal deficit impulse on economic growth depends on the level of the initial debt to capital (or income) ratio, which may reproduce stylized facts.

$^5$ Futagami et al. (1993) introduce public capital as a stock and show that this gives rise to transitional dynamics, in contrast to Barro model with flow expenditures. Introducing public debt also generates transitional dynamics, so we do not need to model public capital (our model is extended to public investment in appendix).
Section one presents the model. Section two deals with steady state issues and shows the multiplicity result. In section three, we discuss the effect of fiscal deficits and taxes in the long run. Section four describes the transitional dynamics of the model, section five depicts the non-linear character of fiscal deficit impulses and section six concludes the paper.

I. The model

We consider a closed economy with a private sector and a government. The private sector consists on a producer-consumer infinitely-lived representative agent, who maximizes the present value of a discounted sum of instantaneous utility functions based on consumption $c_t > 0$, with $\rho > 0$ the discount rate and $S \equiv -u'_c c_t / u_c > 0$ (with $u_c \equiv du(c_t)/dc_t$) the consumption elasticity of substitution:

$$U = \int_0^\infty u(c_t) \exp(-\rho t) dt, \quad \text{with } u(c_t) = \begin{cases} -S & \text{for } S \neq 1 \\ \log(c_t) & \text{for } S = 1 \end{cases}$$ (1)

For the intertemporal utility $U$ to be bounded, we also have to ensure that $(S-1)\gamma_c < S\rho$, with $\gamma_c$ the long-run growth rate of the variable $x$.

As a producer, the representative agent generates per capita output $y_t$ using per capita private capital $k_t$ and per capita productive public expenditures $g_t$. Population is normalized to unity and $0 < \alpha < 1$ is the elasticity of output to private capital:

$$y_t = k_t^\alpha g_t^{1-\alpha}$$ (2)

Public expenditures $g_t$ enter as a flow in the production function and no congestion is present, so that (2) is comparable to the production function of Barro (1990).

The representative agent budget constraint is (we define $\dot{x} \equiv dx_t / dt, \forall x_t$):

$$\dot{k}_t + \dot{b}_t = rb_t + (1-\tau) y_t - c_t - \delta k_t$$ (3)

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6 This condition corresponds to a no-Ponzi game constraint $\gamma_c < r$, with $r$ the real interest rate to be defined.

7 See Fisher & Turnovsky (1998) for a comprehensive study of congestion phenomena in public investment.

8 The condition $0 < \alpha < 1$ ensures the existence of a competitive equilibrium, since, at the representative agent level, $g_t$ is exogenous and the production function exhibits decreasing returns to scale. In equilibrium, on the contrary, $g_t$ is endogenously determined and the production function exhibits constant returns to scale, a necessary condition for a constant growth path to appear in the long run. Barro & Sala-i-Martin (1992) and Turnovsky (1995) discuss the issue of government expenditures in the production function and Aschauer (1989) provides empirical evidence. Appendix A extends our results to a model with public investment, as in Futagami et al. (1993).
Households use their income \((y_t)\) to consume \((c_t)\), invest \((\dot{k}_t + \delta k_t)\), with \(\delta\) the rate of private capital depreciation, and to pay taxes (as in Barro, 1990, we assume a flat tax rate on output \(\tau\)). Households can buy government bonds \((b_t)\), which return the real interest rate \(r_t\). In equilibrium, \(r_t\) equals the marginal productivity of private capital.

The government taxes output, provides productive public expenditures and borrows from households whenever current spending exceeds current income, in which case he will have to pay interests on the issued debt:  

\[
\dot{b}_t = r_t b_t + g_t - \tau y_t
\]

Notice that (4) is an extension of the Barro (1990) government constraint \(g_t = \tau y_t\). In our model, as suggested by data, we introduce public deficits, so that government can make productive expenditures eventually higher than fiscal revenues \(\tau y_t\). However, public deficits generate unproductive public expenditures, in the form of interest payments on debt, which may crowd out productive expenditures in the government budget constraint.

The maximization of (1) subject to (2)-(3)-(4), \(k_0\) given and the transversality condition \(\lim_{t \to \infty} \exp \left( - \int_0^t r_s ds \right) (k_t + b_t) = 0\), gives rise to the familiar Keynes-Ramsey relation:

\[
\gamma_c = \frac{\dot{c}}{c} = S \left[ \alpha (1 - \tau) \left( \frac{g}{k} \right)^{1-\alpha} - \delta - \rho \right]
\]

which describes the rate of growth of consumption (time indexes will be henceforth omitted).

Next, we write (3) and (4) in a more convenient form by displaying the rate of growth of private capital (the IS equilibrium) and of public debt:

\[
\gamma_k = \frac{\dot{k}}{k} = \left( g/k \right)^{1-\alpha} - c/k - g/k - \delta
\]

\[
\gamma_b = \frac{\dot{b}}{b} = \alpha (1 - \tau) \left( g/k \right)^{1-\alpha} - \tau \left( g/k \right)^{1-\alpha} - \delta - \frac{g/k}{b/k} + \frac{g/k}{b/k}
\]

II. The multiplicity of steady state rates of economic growth

Relations (5)-(6)-(7) form a system with three equations for four variables, thus a free variable remains in the government budget constraint (7). To reproduce persistent fiscal

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\(^9\) Taxing interest revenues does not affect our model, except for dual variables in the maximization program.

\(^{10}\) Of course, Ricardian Equivalence does not hold in our model with productive public spending and distortionary taxes. Furthermore, we abstract from seigniorage revenues for the Government (see Minea & Villieu, 2009b).
deficits, we specify a deficit rule, namely that the ratio of deficit to output is equal to some value $m$:\footnote{From a technical point of view, $m$ has to be a constant in steady-state only. Our results would be qualitatively unchanged if we considered a simple rule like $\bar{m}_t = \xi (m_t - m^*)$, with $m^* \geq 0$ the steady-state deficit ratio and $\xi < 0$ the adjustment speed, for example.}

$$\frac{\bar{b}}{y} = m$$ \hspace{1cm} (8)

An endogenous growth solution is defined as a constant long-run growth rate of a variable. Looking (for example) at the consumption growth rate, we can see that for a given $g$, the quotient $g/k$ is decreasing in $k$. This means that the growth rate of consumption in (5) comes to zero in the long run – no growth is present. For this scenario not to happen, a continuous growth of public expenditures $g$ is needed, so that the ratio $g/k$ remains constant in the long run. If this is the case, consumption grows continuously, at a constant rate. For example, in Barro (1990), the balanced budget hypothesis ($g = \tau y$) ensures that $g/k$ is constant in the long run. In this model, we proceed in a similar way, except that the $g/k$ ratio will depend on $\tau$ and $m$ in the long-run.\footnote{Thus, as in Barro (1990), the $g/k$ ratio is endogenous in our model (notice that if this ratio was exogenously chosen by the government, the long run rate of economic growth in the Keynes-Ramsey relationship (5) would also become exogenous, and no endogenous growth path would emerge). The fact that productive expenditures are the adjustment variable in the government budget constraint is well documented in, e.g. Alesina & Perotti (1997) or Martin & Oxley (1991, p.161): "Most countries have offset such increases by winding back public investment, reflecting the political reality that it is easier to cut-back or postpone investment spending than it is to cut current expenditures".}

In what follows, we are interested in the effect of deficit and tax rules on economic growth and welfare in a competitive equilibrium, in a restricted class of models with constant deficit and tax rules $m$ and $\tau$.\footnote{Thus, we do not look for an optimal tax $\{\tau_t\}_0^\infty$ or deficit $\{m_t\}_0^\infty$ paths in a "Ramsey-problem" for a benevolent government, as do, for example, Jones, Manuelli & Rossi (1993). Notice that none of the optimal plans in Jones et al. (1993) conducts to deficit in the long-run, while we particularly care for this stylized fact.} Thus, we must wonder about what relevant policy rules we must enforce in a long-run perspective. Clearly, a constant ratio of tax on national income ($\tau$) is a good premise for studying long-run economic growth, as does Barro (1990). What about deficit rules? To our knowledge and as acknowledged in the introduction, only few papers break with the balanced-budget hypothesis. Yet a quick inspection of OECD data described in the introduction shows that positive deficits are the rule, since for three decades the average ratio of fiscal deficit in these countries was around 2.5%. Thus, a constant deficit rule such as: $m \geq 0$ (for example: $m = 2.5\%$ ), is consistent with OECD data on the long run, while a
balanced-budget rule \((m = 0)\) hypothesis would not be. In addition, some countries, like EMU’s ones, have passed fiscal rules like (8).

Using the debt evolution rule (4), we can write (8) in a more convenient way:

\[
g/k = (m + \tau)(g/k)^{1-a} - [\alpha(1-\tau)(g/k)^{1-a} - \delta]f(b/k) \tag{9}
\]

To search for endogenous growth solutions we define the three modified variables:

\[
c_k \equiv c/k, \quad g_k \equiv g/k \quad \text{and} \quad b_k \equiv b/k.
\]

Equations (5)-(6)-(7) and (9) become:

\[
\begin{align*}
\text{(a)} & \quad \frac{\dot{c}_k}{c_k} = \frac{\dot{c}}{c} = S \left[ \alpha(1-\tau)(g_k(b_k))^{1-a} - \delta - \rho \right] + g_k(b_k) + c_k + \delta - \left( g_k(b_k) \right)^{1-a} \\
\text{(b)} & \quad \frac{\dot{b}_k}{b_k} = \frac{\dot{b}}{b} = \frac{m(g_k(b_k))^{1-a}}{b_k} + g_k(b_k) + c_k + \delta - \left( g_k(b_k) \right)^{1-a} \\
\text{(c)} & \quad g_k = (m + \tau)g_k^{1-a} - rb_k, \quad \text{with} \quad r = \alpha(1-\tau)g_k^{1-a} - \delta
\end{align*}
\tag{10}
\]

We find the steady state endogenous growth solutions by imposing \(\dot{c}_k = \dot{b}_k = 0\). Together with the third equation, we obtain constant values of \(c_k, b_k\) and \(g_k\), which means that the four initial variables (\(c, k, g\) and \(b\)) grow at the same constant rate \(\gamma\). Putting \(\dot{c}_k = \dot{b}_k = 0\), we eliminate \(c_k\) by extracting (10b) from (10a). Reintroducing the obtained expression of \(b_k\) in (10c) provides an implicit expression in \(g_k\) (note that \(\gamma\) is a (increasing) function of \(g_k\) in (5)), with parameters \(S, \alpha, \delta, \rho, \tau\) and \(m\):

\[
F(g_k) = \gamma(g_k)((m + \tau) - g_k^\alpha) - m \left( \frac{\gamma(g_k)}{S} + \rho \right) = 0 \tag{11}
\]

Since \(\gamma = \gamma_c\) in steady state, we can write (11) as a system:

\[
\begin{align*}
\text{(a)} & \quad \gamma'(g_k) = S \left[ \alpha(1-\tau)g_k^{1-a} - \delta - \rho \right] \\
\text{(b)} & \quad \gamma^2(g_k) = m\rho \left[ m \left( \frac{S-1}{S} \right) + \tau - g_k^\alpha \right] + m \left( \frac{S-1}{S} \right) + \tau
\end{align*}
\tag{12}
\]

**Proposition 1:** Multiplicity of long-run rates of economic growth

(A) if \(m = 0\), the system (12) has two solutions. The first is the "Solow solution" corresponding to a zero long run growth rate, and the second is the "Barro solution";

(B) if \(m > 0\), one can find a set of admissible parameters \(S, \alpha, \delta, \rho, \tau, m\), so that the system (12) has two solutions leading to positive long run growth rates;
(C) if \( m < 0 \), the system (12) has once again two solutions, but one only leads to a positive long run rate of economic growth, the other being negative. In this case, the positive long run growth rate is associated with a negative stock of public debt.

Proof: see Fig. 1 for \( S = 1 \) and Appendix 1 for \( S \neq 1 \). Appendix 2 provides approximate analytical values of the two balanced rates of economic growth.

Fig. 1 describes the steady-state rates of economic growth for \( S = 1 \). The \( \gamma^2(g_k) \) curve is a hyperbola, with asymptote \( g_k = \tau^{1/\alpha} \). For \( m > 0 \), the \( \gamma^2(g_k) \) curve is the continuous line plotted in Fig. 1, while it is the dotted line for \( m < 0 \). The steady state solutions are defined as the points of intersection between the two curves \( \gamma^1(g_k) \) and \( \gamma^2(g_k) \). If \( g_k > (m + \tau)^{1/\alpha} \), the steady state stock of public debt is negative in (12). Thus, for \( m < 0 \), one solution (\( \bar{L} \) point) leads to a negative long run rate of economic growth, which we exclude, and the other (\( \bar{H} \) point) leads to a negative stock of public debt (because \( g_k > \tau^{1/\alpha} > (m + \tau)^{1/\alpha} \)). Since, governments do not have a creditor position in the long run, we restrict our analysis to cases (A) and (B), corresponding to \( m \geq 0 \).

The (A) case \( (m = 0) \)

A balanced budget rule means that tax revenues finance productive public expenditures and debt interest payments. Notice that this case does not perfectly correspond to the model of Barro (1990); the difference is that, while in Barro (1990) public debt is always at zero, in our scheme public debt may be positive. This slight change has important consequences, because two steady state solutions emerge.

Putting \( m = 0 \) eventually makes the \( \gamma^2 \) curve (12b) undetermined\(^{14} \), but we easily find the two solutions from (11). The first solution is the Barro (1990) one, for \( g_k^B = \tau^{1/\alpha} \):

\[
\gamma^B = S \left[ \alpha(1 - \tau)(g_k^B)^{1-\alpha} - \delta - \rho \right] > 0.
\]

This solution is the \( B \) point in Fig. 1 and corresponds to a zero stock of public debt in steady state \( b_k^B = 0 \).

\(^{14}\) If \( m = 0 \) and \( g_k^\theta = \tau \), (12b) produces a 0/0 case of indeterminacy. Thus, for \( m = 0 \), the \( \gamma^2 \) curve is either \( \gamma = 0 \) or \( g_k = \tau^{1/\alpha} \), as in Fig. 1.
The second solution corresponds to a zero growth steady state $\gamma^s = 0$, with

$$g^s_k = \left( \frac{\delta + \rho}{\alpha (1 - \tau)} \right)^{\frac{1}{1-a}}.$$ 

We call this steady state the “Solow solution” (S point of Fig. 1), which is reached for a positive stock of public debt (from (10c)): $b^s_k = \frac{(g^s_k)^{1-a}}{\rho} \left[ \tau - (g^s_k)^a \right] > 0$. The existence of this second solution is due to the fact that the constant level of public debt forces private capital to be constant in the long-run to achieve a constant steady-state $b^s_k$ ratio. Thus, economic growth disappears in the long-run ($\gamma^s = 0$).

The intuition behind this no-growth solution is that, contrary to Barro (1990), not all public expenditures are productive in the steady-state, if public debt is positive. Interest payments on debt generate non-productive public expenditures and crowd-out productive expenditures. Since in the steady-state tax revenues must finance productive expenditures plus the interest payments on public debt, $b^s_k$ corresponds to a level of debt such that interest payments absorb a great part of government resources: remaining resources for productive expenditures cannot generate a positive rate of economic growth, which explains why a “Solow solution” emerges in our model.

The (B) case ($m > 0$)

With persistent deficits, three cases may occur. The most interesting of them, illustrated in Fig. 1, exhibits two points of intersection between $\gamma^1$ and $\gamma^2$, namely $L$ and $H$, both leading to endogenous positive long run growth rates (further called the low and the high growth rates, $\gamma^L$ and $\gamma^H$). By changing parameters, one can obtain a unique solution for the system (12) or even no solution, when $\gamma^1$ passes below $\gamma^2$. Approximate analytical values of the two balanced rates of economic growth are given in Appendix 2.

Let us try an intuitive interpretation of the (B) case (when multiplicity occurs), compared to the (A) case. With a deficit rule ($m > 0$), public debt expands at a positive stationary growth rate ($\dot{b}/b = my/b$), forcing the long run growth rate of the economy to move away from the Solow solution (the low steady state solution moves from $\gamma^s$ to $\gamma^1$). Similarly, introducing deficit generates a positive level of public debt in the long run, which

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15 Multiplicity refers to the existence of a finite number of solutions, and indeterminacy stands for the case in which the number of solutions is infinite or in which we cannot select a determinate solution or trajectory.
requires interest payments. Therefore the Barro solution can no longer be reached. In this case, the high steady state solution goes from $\gamma^b$ to $\gamma^H$.

**Fig. 1 – Steady state rates of economic growth**

III. Fiscal deficits and taxes in the long run

In this section, we study comparative statics in response to a change in the deficit ratio or in the tax rate. Notice that we are not interested in the comparison between taxes and deficit financing, 16 but in deriving the effects of a *ceteris paribus* change in one instrument, with the other (and all other parameters) given. Our results are synthesized in *Propositions 2 and 3* below. *A priori* no statement can be made about the effect of a tax rate or a fiscal deficit impulse, because of multiplicity. However, as we shall see in *Section IV* below, the low solution of system (12) is unstable; therefore no further interest is given to this solution for the moment. Let us focus on the high solution.

**Proposition 2: Effect of an increase in the deficit ratio ($m$)**

*a) Any ceteris paribus (in particular, the same tax rate) increase in the deficit-to-GDP ratio lowers the high-rate of economic growth in the long-run.*

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16 In particular because, as we will see below, permanent budget deficits are not a way of government finance in the long run, but on the contrary generate new expenditures (the public debt burden).
b) It is impossible for the long-run growth rate to exceed the Barro solution, obtained with a balanced-budget rule (and no public debt in steady-state), which is the highest long-run rate of economic growth.

Proof: From (12b): \[
\frac{d \gamma^2}{dm} \bigg|_{g_k \text{ given}} = \frac{\rho \left( \tau - g_k^a \right)}{m \left( \frac{S-1}{S} \right) + \tau - g_k^a} \geq 0, \text{ if } m \geq 0. \]

Thus, an increase in the deficit ratio moves the \( \gamma^2 \) curve towards the left. Since \( \gamma \) and \( g_k \) are positively linked in the \( \gamma^1 \) curve, we have: \( \gamma^{\mu} < \gamma^{\mu} \). Notice in particular that: \( g_k^{\mu} < g_k^{\nu} = \tau^{1/\alpha} \).

As depicted by Fig. 2 below, any increase in the deficit ratio raises the (unstable) low economic growth rate (dotted line) and reduces the high economic growth rate (continuous line). The highest acceptable value for the deficit ratio \( (\bar{m}) \) is computed in Appendix 2.

Fig. 2 – The effect of the deficit ratio \( (m) \) on long-run economic growth \( (\gamma) \)

Proposition 2 shows that public deficits *ceteris paribus* reduce growth, if we are exclusively concerned with *long-run stable solutions*. Any deficit rule \( m > 0 \) never allows the rate of economic growth to exceed the Barro solution \( (m = 0) \), even if this deficit is initially fully designed for productive public expenditures. Our result is due to the transversality condition, which constraints the debt growth rate to be lower than the real interest rate in the long-run. Since, in the long-run, all variables grow at the same rate, the transversality condition \( \gamma < r \) means that \( b/b < r \), and further that \( my < rb \), namely that unproductive public expenditures (interest payments) exceed deficit revenues. Thus, in the government budget constraint, the debt service necessarily exceeds the new resources provided by the
deficit rule in the steady-state, generating a crowding-out effect on productive public spending: \( g/y < \tau \) in equation (4) because \( my < rb \). Finally, notice that this result is very general and does not depend on the type of public spending (flow or stock, see Appendix A), the presence or absence of unproductive public spending (utility-enhancing or wasteful) or the definition of the deficit rule (see Minea & Villieu, 2009a). Proposition 2 simply comes from the fact that fiscal deficits are not a way of government finance in the long-run: the permanent flow of new resources from deficits \( (b) \) is always lower than the permanent flow of new expenditures resulting from the debt burden \( (rb) \) in steady-state.

Compared to the effects of fiscal deficits, the effects of a tax-rate impulse are more delicate to assess, since a higher tax-rate moves downward the two curves \( \gamma^1 \) and \( \gamma^2 \). We obtain the following results.

Proposition 3: Effect of an increase in the tax-rate \( (\tau) \)

a) There exists one value of the tax-rate that maximizes the high-rate of economic growth. If \( m = 0 \) this value is, as in Barro (1990): \( \tau^h = 1 - \alpha \). If \( m > 0 \), the high-rate of economic growth reaches its maximum at \( \tau^* > \tau^h \).

b) The maximizing-growth tax rate \( \tau^* \) is an increasing function of the deficit ratio.

Proof: We rewrite equation (11) as a function of \( \gamma \) and \( m \), and use the definition of \( g_k \) in

\[
(12a): F(\gamma, m) = \gamma \left( m + \tau \right) - \left[ \frac{\gamma + S(\delta + \rho)}{\alpha S(1 - \tau)} \right]^\frac{1}{1 - \alpha} - m \left( \frac{\gamma + \rho}{S} \right) = 0. \]

The first order condition for \( \tau \) to be an optimum is, from the implicit function theorem:

\[
\tau^* = 1 - \alpha + \alpha m \left( \frac{\rho}{\gamma(\tau^*)} \right) - \left( \frac{S - 1}{S} \right) \geq 1 - \alpha
\]

Of course, (13) gives only an implicit form for \( \tau^* \) (Appendix 3 derives an explicit analytical value of \( \tau^* \)). Nevertheless, for the solvability condition to be enforced, we have: \( \rho / \gamma(\tau^*) - (S - 1)/S > 0 \); thus, the maximizing-growth tax rate \( \tau^* \) is higher than the Barro solution \( \tau^h = 1 - \alpha \), for any positive level of the deficit ratio. Similarly to Barro (1990), flat-rate taxes have a positive growth effect, by providing resources for growth-enhancing public spending, and a negative effect, because they distort the private capital accumulation. Notice that we find the Barro solution as a special case if \( m = 0 \).
The economic interpretation of Proposition 3 is the following. Let $\tau^*$ be the tax rate that maximizes long-run growth. From (7), the elasticity of $g_y = g / y$ with respect to this tax rate is defined by:

$$
\frac{dg_y / g_y}{d \tau / \tau} = \frac{\alpha}{1 - \alpha} \left( \frac{\tau^*}{1 - \tau^*} \right) = \varepsilon_{\tau^*}(\tau^*)
$$

(14)

In Barro (1990), the balanced-budget rule imposes that: $g_y = \tau$, so that the elasticity of the ratio of public spending to output with respect to the tax-rate is: $\varepsilon_{\tau^*}^y = 1$. Equalizing (14) to this elasticity yields the Barro solution: $\tau^b = 1 - \alpha$. In our model, from the government budget constraint we have: $xg_y = \tau$, where $x = \frac{g_k + (r - \gamma)b_k}{g_k}$, so that: $\varepsilon_{\tau^*}^y = 1 - \frac{dx / x}{d \tau / \tau}$. What Proposition 3 shows is that the elasticity of public spending (in % of output) with respect to the tax rate is higher than one $\varepsilon_{\tau^*}^y(\tau^*) > 1$, which, combined with (14), means that $\tau^* > 1 - \alpha$.\(^{17}\)

In other words, contrary to Barro (1990), one extra percent of tax revenues yields more than one extra percent of productive government expenditures, because it enables to decrease the (net-of-deficit resources) burden of public debt service, explaining why in our model the growth-maximizing tax-rate exceeds the Barro one.

In addition, relation (14) shows that the optimal tax-rate is an increasing function of the deficit ratio. Effectively, for small values of $m$ (Appendix 3 extends this result to any $m$-value), we obtain from (14): $\frac{d\tau^*}{dm}\bigg|_{m=0} = \alpha \left( \frac{\rho}{\gamma(\tau)} - \frac{S - 1}{S} \right) > 0$. The intuitive explanation of this result is the following. Suppose that the starting point is the Barro long-run equilibrium with $\tau^b = 1 - \alpha$ and a zero deficit. If $m$ jumps to some positive value, long-run economic growth decreases, since the public-debt burden crowds-out productive spending. To restore (part of) productive public spending, government must increase the tax rate beyond the Barro value.

Proposition 3 shows that, with a deficit rule ($m > 0$), the optimal size of the public sector is larger than under a balanced budget rule. This result is compatible with Proposition 2, stating that the economic growth rate is lower than Barro’s one, as we can see in Fig.3.

\(^{17}\) To illustrate this point, suppose that $(r - \gamma)b$ is independent of $\tau$. Since $dg_y / d\tau > 0$ and using the transversality condition $(r > \gamma)$, we find $-(dx / x)(d\tau / \tau) > 0$. Of course, $(r - \gamma)b$ is not independent of $\tau$; see Appendix 3 for a general proof.
If $\tau > \tau^*$, any increase in the tax rate or in the deficit ratio lowers long-run economic growth. In this case, the two policy instruments are substitutes: one can reach the same rate of economic growth by increasing (respectively decreasing) $\tau$ and decreasing (respectively raising) $m$. If $\tau < \tau^*$, on the contrary, the two instruments are complements: to reach an unchanged rate of economic growth, one must move the tax rate and the deficit ratio in the same direction, since higher taxes increase long-run growth, while higher deficits decrease it.

IV. Transitional dynamics

In addition to the previous steady-state results, our model exhibits non-trivial transitory dynamics. In this section we discuss the impact of deficit rules along the transition path. As we shall see, contrary to their negative effect on long-run growth, deficit-financed increases in productive expenditures may improve economic growth in the short-run.

To check for the stability of the two equilibria, we examine the transitional dynamics of variables $(c_k, b_k)$ in system (10). First, we derive some analytical results from the case $m = 0$; then we offer a graphical representation for the case $m > 0$. Our analysis is supplemented with numerical simulations.

A balanced budget rule ($m = 0$)

For $m = 0$ (no deficit) we find the two steady states expounded in the previous section. First, the Barro steady state is reached for: $b_k = b^B_k = 0$, so that $\dot{b}_k = 0$ in (10b). When public debt is zero, productive public expenditures are: $g_k = g^B_k = \tau^{1/\alpha}$, and the associated steady consumption-to-capital ratio is: $c_k = c^B_k = \left(g^B_k\right)^{-1/\alpha} - \gamma^B - g^B_k - \delta$, which makes $\dot{c}_k = 0$. 

Fig. 3 – The tax-rate ($\tau$) and the high rate of economic growth ($\gamma^H$)
in (10a). Second, we find the Solow steady state for \( g_k = g_k^S \), thus \( b_k = b_k^S \) (both values are defined above), and \( c_k = c_k^S = (g_k^S)^{-\alpha} - g_k^S - \delta \), so that \( \dot{c}_k = \dot{b}_k = 0 \) with zero long run growth.

Graphically, the stability locus of \( c_k \) \((\dot{c}_k = 0)\) describes an inverted-U curve with a maximum at \( \hat{b}_k \), as represented in Fig.4. In the same chart we plot the stability locus of \( b_k \) \((\hat{b}_k = 0)\), which consists of two curves: the vertical curve \( b_k = b_k^B = 0 \) and a decreasing curve in \( b_k \), namely \( c_k = (g_k(b_k))^{-\alpha} - g_k(b_k) - \delta \). The \( B \) point is the “Barro solution” and the \( S \) point is the “Solow solution” (no growth); on the right hand of \( b_k^S \) economic growth is negative, while positive on the left hand side. (10) shows that the \( \dot{b}_k = 0 \) curve is always above the \( \dot{c}_k = 0 \) curve, except when the long run rate of growth is zero (point \( S \)) or negative (on the right hand of \( b_k^S \)). Since \( b_k \) is predetermined and \( c_k \) jumps, the dynamics clearly show that \( B \) is a saddle point, while \( S \) is unstable, as confirmed by analytical results in Appendix 4.

Fig. 4 – Phase diagram in the case of a balanced budget rule \((m = 0)\)

A deficit rule \((m > 0)\)

When the government runs into debt \((m > 0)\), analytical results are delicate to obtain. We can find a set of admissible parameters, so that the phase diagram changes slightly with
respect to the case $m = 0$. The major changes concern the $\dot{b}_k = 0$ locus, which now has an inverted-U shape (with a maximum at $\tilde{b}_k$) and presents an asymptote at $b_k = b_k^B = 0$, so that the $B$ steady state can no longer be reached. Moreover, recall from the previous section that the two steady states $L$ and $H$ are respectively on the left and on the right of $S$ and $B$.

**Fig. 5 – Phase diagram in the case of a deficit rule ($m > 0$)**

We construct the steady state curve for different values of the deficit ratio by extracting $m$ from the $\dot{c}_k = 0$ relation and substituting it into the $\dot{b}_k = 0$ relation. Numerical simulations show that this steady state locus takes the form of an increasing curve, represented by the $SS$ curve in Fig.5 (for $m \geq 0$). The two extreme points of this curve are the Barro steady state ($B$ point, for $\tilde{b}_k = 0$), and the Solow steady state ($S$ point, associated

\[ g(\tilde{b}_k) = (1 - \alpha)^{1/\sigma} > g(\hat{b}_k) \text{ if } S < 2/\alpha(1 - \tau) \] which we suppose, and $\tilde{b}_k$ (i.e. the maximum of the curve $\dot{c}_k = 0$) by:

\[ g(\hat{b}_k) = ((1 - \alpha)(\alpha(1 - \tau)S - 1))^{1/\sigma} \] In Fig.4, we considered (without generality loss) a representation in which $\tilde{b}_k = 0 - b_k^B$ (this is obtained for $\tau = \tau^B = 1 - \alpha$), so that $\tilde{b}_k < b_k^S < \hat{b}_k$.

---

18 Recall that $m$ is small, ordinary less than 5%. Appendix 4 presents simulations on local stability.

19 $\tilde{b}_k$ is implicitly defined by: $g(\hat{b}_k) = (1 - \alpha)^{1/\sigma} > g(\tilde{b}_k)$ if $S < 2/\alpha(1 - \tau)$ which we suppose, and $\hat{b}_k$ (i.e. the maximum of the curve $\dot{c}_k = 0$) by: $g(\hat{b}_k) = ((1 - \alpha)(\alpha(1 - \tau)S - 1))^{1/\sigma}$. In Fig.4, we considered (without generality loss) a representation in which $\tilde{b}_k = 0 - b_k^B$ (this is obtained for $\tau = \tau^B = 1 - \alpha$), so that $\tilde{b}_k < b_k^S < \hat{b}_k$.

16
to $b_k^s$ and no growth). These two extrema correspond to a balanced budget rule ($m = 0$), while steady-state points located inside the segment $BS$ (like points $L$ and $H$) correspond to a deficit rule ($m > 0$); the multiplicity of steady states implies that each value of $m$ is related to two points on the $SS$ curve (up until $\bar{m}$).

The phase diagram (Fig. 5) points out that, similarly to the $m = 0$ case, the high steady state ($H$) is a saddle point and the low steady state ($L$) is unstable (recall that $b_k$ is predetermined and $c_k$ jumps). Fig. 5 also shows that the paths of variables turn around $L$, following an unstable spiral. Thus, an economy starting with a very high debt stock $b_k > \bar{b}_k$ cannot reach any steady state; in this case, there is no initial jump of $c_k$ that puts the economy on a saddle path. To avoid such unstable paths, in which the public debt-to-capital (or output) ratio explodes, we impose a superior bound for $b_k$ ($b_k \leq b_k^{\text{max}}$), which in turn defines a certain maximum “sustainable” level of public debt, namely the highest debt ratio that the society is prepared to finance.\textsuperscript{20}

Consequently, at some date $T$ (to be endogenously determined), the debt-to-capital ratio will reach $b_k^{\text{max}}$, and at this point in time the government must adopt the deficit rule $m$ which makes $T$ a steady state. In what follows, we define $b_k^{\text{max}}$ as the highest debt-to-capital ratio that is consistent with the existence of a steady state; in Fig. 5 this ratio corresponds to the Solow solution ($b_k^{\text{max}} = b_k^s$). Thus, the steady deficit rule that must be adopted at $T$ is $m = 0$. In this case, public debt is seen as “sustainable” as long as $\gamma \geq 0$.

This set-up produces indeterminacy. In our model, indeterminacy has two meanings. First, it takes the form of multiple (two) balanced long run growth paths: an endogenous growth path ($H$ solution) and a “poverty trap” without growth ($S$ solution), obtained when the public debt sustainability condition is reached.\textsuperscript{21} Second, indeterminacy denotes that one cannot choose between the multiple (two) transition paths to steady state equilibrium. Fig. 5 exhibits three kinds of solutions, depending on the initial public-debt-to-capital ratio. Ceteris paribus, an economy endowed with a low initial public debt ratio ($b_{k_0} < \bar{b}_k$) converges to the high balanced growth path ($H$), while an economy endowed with a high initial public debt

\textsuperscript{20} The sustainability condition points out the social and political bound on public borrowing, and has to be distinguished from the solvency condition $\gamma < r$. Such an upper bound is often assumed when the public debt path is unstable (see, e.g. Sargent & Wallace, 1981).

\textsuperscript{21} Of course, if the sustainability condition is $b_k^{\text{max}} < b_k^s$ the poverty trap means a “slow” balanced growth path.
ratio \( b_{k0} > \bar{b}_k \) converges to the “poverty trap” solution \( S \).\(^{22}\) But for economies endowed with an intermediate initial public debt ratio \( \bar{b}_k < b_{k0} < \bar{b}_k \), the transition path is radically indeterminate. These economies may reach either the convergent path to the high equilibrium \( H \) (the consumption-to-capital ratio jumps down), or the path leading to the “no growth trap” \( S \) (if the consumption-to-capital ratio jumps up). This situation reminds of the “history versus expectations” scenario of Krugman (1991): for countries with an initial high or low public debt, “history”, in the form of the initial public debt ratio, determines the equilibrium path, while for countries with an intermediate initial public debt ratio, the transition path is subjected to “expectations”, in the form of “optimistic” or “pessimistic” views on debt sustainability.

V. Non-linear effects of fiscal deficit on welfare and economic growth

Suppose now an upward jump of the deficit-to-output ratio, from an initial steady state position \( m \) to the new steady state described by the deficit ratio \( m' \) \( (m' > m) \). Thus, the two steady states \( (L \) and \( H) \) shift respectively to \( L' \) and \( H' \) on the SS curve, as in Fig.6.

Fig. 6 – Transitional dynamics following a positive impulse on deficit ( \( m \) increases)

\[^{22}\] In these two cases, the saddle path is reached by an initial jump of \( c_{k0} \), which rules out any divergent path.
If the economy starts from the $H$ steady state, for a predetermined public-debt-to-capital ratio $\left( b_{k_0} = b_{k}^H \right)$, $g_{k_0}$ jumps up in (10c) and the rate of economic growth increases initially. Consequently, contrary to our steady state result, debt-financed productive expenditures may lead to a higher economic growth rate in the short run.\textsuperscript{23} We describe in Fig.6 initial adjustments and transitional dynamics, following a positive impulse on the deficit ratio, with initial steady states $L$ or $H$, in the spirit of Fig.5.

If the economy is initially in $H$, we can either have a negative (as in Fig.6) or a positive jump in initial consumption, with a lower growth rate in the steady state $H'$. However, if the economy initially starts from the low steady state $L$, indeterminacy occurs. For a predetermined public-debt-to-capital ratio $\left( b_{k_0} = b_{k}^L \right)$, the initial consumption-to-capital ratio can either jump upward (to $c_{t_0}^3$), if agents anticipate that public debt will become unsustainable at some future date $T$, or jump downward (to $c_{t_0}^2$), if agents are “optimistic” about public debt sustainability. In both cases, economic growth increases initially, but, because of the collapse of the consumption-to-capital ratio, it increases much more in the second case. In Table 2, the rate of economic growth jumps from $\gamma^L = 0.8\%$ to $\gamma^3_{t_0} = 1.81\%$ in the second case, but only to $\gamma^3_{t_0} = 0.84\%$ in the first case, with capital accumulation going through similar changes.

\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|c|c|c|c|c|c|c|}
\hline
\textbf{()} & $b_k$ & $c_k$ & $g_k$ & $\gamma_y$ & $\gamma_k$ & $\gamma_c$ & $\gamma_{b_k}$ & $\gamma_{s_k}$ \\
\hline
\textbf{Initial Steady State} & 53.57 & 25.01 & 11.97 & $\gamma^L = 0.80$ & & & & 0 \\
\hline
\textbf{Initial jump of variables} & & & & & & & & \\
\textbf{(case 2, case 3)} & 53.57 & 24.28 & 12.60 & 1.81 & 1.78 & 1.44 & -0.15 & 0.00 \\
\hline
\textbf{Final Steady State} & & & & & & & & \\
\textbf{(H',S)} & 13.35 & 19.44 & 21.52 & $\gamma^L = 8.13$ & & & & 0 \\
\hline
\end{tabular}
\caption{Simulation results for an increase in the deficit-to-GDP ratio from $m = 1\%$ to $m = 2\%$ – Starting from $L$ Steady State}
\end{table}

For simulation values: $\alpha = 0.6\, , \, \rho = 0.1\, , \, S = 2\, , \, \tau = 0.4\, , \, \delta = 0.05$

These initial jumps in consumption and in the growth rate of output or capital accumulation determine the transition path that will be reached by the economy. With a downward jump of consumption, the economy is able to produce a sufficiently high rate of capital accumulation to reduce the debt-to-capital ratio. Consequently, all variables converge to the high steady state $H'$, with declining public debt ratio and self-enforcing economic growth. With an upward jump in consumption, on the contrary, the initial jump of economic

\textsuperscript{23} This is the case if the economy initially departs from the $B$ steady state. In other words, in the short run, the rate of economic growth may exceed the Barro solution.
growth does not allow public debt to decline; the economy moves to the “poverty trap” (the $S$ point), which it reached at time $T$.

Fig.7 – Non-linear effects of fiscal deficit on economic growth

Our model reveals the non-linear characteristic of a fiscal deficit impulse in the medium and long run. In all three cases of Fig.7, public deficit is positively linked to economic growth in the short run. On the contrary, in the medium or long run, fiscal deficit may be positively or negatively related to economic growth, with an unchanged set of parameters, according to the initial public-debt-to-capital ratio.

Growth and deficits are negatively related in the neighborhood of the high steady state, where the public-debt-to-capital ratio is small (case 1). In the neighborhood of the low steady state, with a high public-debt-to-capital ratio, the relation between economic growth and fiscal deficit depends on expectations. If the fiscal policy stance is seen to become unsustainable at some future date, the economy converges to the no-growth solution: once more, economic growth and fiscal deficits are negatively related (case 3). If agents are “optimistic”, however, the initial cut in consumption is enough to generate a permanent positive growth effect and the economy converges to the high steady state. In this case, the link between public deficit and economic growth is positive (case 2). Thus, our model can reproduce the so-called “Keynesian” or “anti-Keynesian” behavior of consumption and economic growth, following a raise in the deficit to GDP ratio. When the public debt ratio is high, an expectation switch

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24 An increase in deficit may also raise long-run economic growth if Government finds a way to “sterilize” the increase it the debt burden engendered by a higher deficit, either by decreasing unproductive public spending (see Minea & Villieu, 2009a) or fixing the long-run debt-to-GDP-ratio (see Futagami et al., 2008). However, in both cases, economic growth is still below the zero-deficit growth rate (namely, the Barro solution).
about its sustainability may reverse the effect of deficits on economic growth, as in Feldstein (1982) or Sutherland (1997), but in a very different set up without uncertainty.25

In the spirit of the empirical work described in introduction, what produces this non-linearity is an expectation effect on the credibility of fiscal policy, when the initial debt-to-capital ratio is very high (Perotti, 1999). If households judge public debt as sustainable, the rise of public deficit is credible and the economy grows to the high solution. On the contrary, if the deficit rise is seen to turn unsustainable, the economy is condemned to a “poverty trap”.

VI. Conclusion

Does higher public deficit, increasing the burden of public debt, generate a crowding out effect on economic growth or does it ensure the development of productive infrastructures which promote long run growth-enhancing? In a very stylized model of endogenous growth, we show that the two effects may occur, depending fundamentally on i) the initial debt-to-capital (or output) ratio and ii) expectations about government debt sustainability. Although analytically simple, our model exhibits some complex dynamics, allowing for non-linear effects of fiscal policy, which may reproduce a number of empirical results.

Our model does not contradict Keynesian views in the short and medium run (a fiscal deficit impulse may increase economic growth), but it does in the long run; consequently, one must be careful in assessing the growth-benefits of fiscal deficits: a debt financed increase in productive expenditures boosts economic growth in the short run, but the burden of repayment will weigh the balanced growth path down in the long run.

Moreover, if the initial fiscal stance is close to equilibrium, allowing for debt-financed productive public expenditures does not ensure to fill a shortage of public investment in the long-run. On the contrary, the shortage of public investment may be the consequence of excessive government borrowing, either for productive or unproductive expenditures. However, if the initial fiscal stance is much damaged, our model produces another effect: a deficit-financed impulse in productive expenditures may (or not) generate self-fulfilling optimistic expectations, which allow the economy to leave the low-growth balanced path and to catch up with the high-growth balanced path. Unfortunately, the alternative between optimistic or pessimistic views on the future is not a matter of choice for the government, and the risk of such a debt policy is to condemn the economy to a no growth trap.

25 Our model produces another kind on non-linearity, namely the asymmetric effects of a cut in deficit. Even if we abstract indeterminacy issues, a deficit cut will have a much bigger impact on the transitory growth path if the economy starts from the \( L \) steady state (high initial debt), than if it starts from point \( H \) (low initial debt).
References

Appendix 1: The steady state rates of economic growth for $S \neq 1$

If $S < 1$, notice first that the asymptote $\hat{g}_k = \left[ m \left( \frac{S-1}{S} \right) + \tau \right]^{1/\alpha}$ is located on the left of $g_k = \tau^{1/\alpha}$, so Proposition 2 trivially holds (this situation is depicted by the dotted lines, and the two steady state are $\hat{L}$ and $\hat{H}$). If $S > 1$, we can see that the $\gamma^2(g_k)$ curve intersects the $g_k = \tau^{1/\alpha}$ line for the highest acceptable rate of economic growth $\gamma^\text{max} = \rho g(S-1)$, subject to the transversality condition. This implies that $\gamma^1(g_k)$ and $\gamma^2(g_k)$ intersect on the left-hand side of $g_k = \tau^{1/\alpha}$, i.e. that the high growth solution $\gamma^H < \gamma^B$. This situation is depicted by the continuous curves, and the two steady states are $\hat{L}$ and $\hat{H}$.

Appendix 2: Analytical value of the steady state growth rates

Equation (11) can be rewritten as:

\[
\left( \frac{\gamma}{S} + \rho + \delta \right) = f(\tau) \left[ 1 - \frac{m}{\tau} \left[ \frac{(1-S)\gamma + S\rho}{S\gamma} \right] \right]^{1-\alpha/\alpha}, \quad \text{or in logarithm}
\]

\[
\log \left( \frac{\gamma}{S} + \rho + \delta \right) = \left( \frac{1-\alpha}{\alpha} \right) \log \left[ 1 - \frac{m}{\tau} \left[ \frac{(1-S)\gamma + S\rho}{S\gamma} \right] \right], \quad \text{where} \quad f(\tau) = \alpha(1-\tau)(\tau^{1-\alpha/\alpha}).
\]
For \( m \to 0 \), we can write: 
\[
\frac{\gamma}{S} + \rho + \delta = \left[ 1 - \left( \frac{1-\alpha}{\alpha} \right) \frac{m(1-\alpha)(1-S)}{\alpha} \right] f(\tau) 
\]
(remark that \( \frac{\gamma}{S} + \rho + \delta \to f(\tau) \) in (A1)), which is a second-degree equation in \( \gamma \): 
\[
\gamma^2 + A\gamma + B = 0, 
\]
where: 
\[
A = S \left\{ \rho + \delta - f(\tau) \left[ 1 - \frac{m(1-\alpha)(1-S)}{\alpha} \right] \right\} \quad \text{and} \quad B = \frac{Sm\rho(1-\alpha)f(\tau)}{\alpha}. 
\]
As the determinant of this second degree equation in \( \gamma \) is positive, it has two real (positive) solutions: 
\[
\gamma_u = \frac{1}{2} \left[ -A + \sqrt{\Delta} \right] \quad \text{and} \quad \gamma_l = \frac{1}{2} \left[ -A - \sqrt{\Delta} \right], \quad \text{where: } \Delta = A^2 - 4B. 
\]
For \( \Delta = 0 \), we find the value of the deficit ratio \( m = \bar{m} \) (depicted in Fig.2) that equalizes the two rates of economic growth (there are two solutions, but only one verifies the transversality condition).

**Appendix 3: Proof of Proposition 3**

From (11), extracting \( g_k \) from (12a) and using the implicit function theorem, we compute: 
\[
\frac{d\gamma}{d\tau} = -\frac{\partial F}{\partial \gamma} \frac{\partial F}{\partial \gamma}, \quad \text{with:} \quad \frac{\partial F}{\partial \gamma} = \frac{\gamma ((1-\tau)(1-\alpha) - \alpha g^*_k)}{(1-\tau)(1-\alpha)}. 
\]
Thus, the tax rate \( \tau^* \) that maximizes economic growth is defined by: 
\[
g^*_k (\tau^*) = [(1-\tau^*)(1-\alpha)/\alpha]^{1/\alpha}. 
\]
Introducing this value in Eq.(12a), we obtain the maximized growth rate, as a function of \( \tau^* \): 
\[
\left( \frac{\gamma}{S} + \rho + \delta \right)^{\alpha} = \left[ \alpha (1-\tau^*) \right]^{\alpha} \left[ g_k(\tau^*) \right]^{1-\alpha} = (1-\tau^*)^{\alpha}, \quad \text{where: } Z = \alpha^{2-1} (1-\alpha)^{-\alpha}, \quad \text{and} \quad \text{from (11):} 
\]
\[
\frac{1-\tau^*}{\alpha} = \left[ 1 + mS - \frac{m\rho}{\gamma^*} \right], \quad \text{thus:} \quad \left[ \frac{\gamma^*}{S} + \rho + \delta \right]^{\alpha} = \left[ 1 + mS - \frac{m\rho}{\gamma^*} \right]^{\alpha}. \]

For \( m \to 0 \), we can write: 
\[
\alpha \left( \frac{\gamma^*}{S} + \frac{\rho + \delta}{\alpha Z^{1/\alpha}} - 1 \right) = mS - \frac{m\rho}{\gamma^*}, \quad \text{which is a second-degree equation in } \gamma^*: \quad \left( \gamma^* \right)^2 + A\gamma^* + B = 0, \quad \text{where:} \quad A = S(\rho + \delta) - S(\alpha Z^{1/\alpha}) - m(S-1)(\alpha Z^{1/\alpha})/\alpha \quad \text{and} \quad B = Sm\rho(\alpha Z^{1/\alpha})/\alpha, \quad \text{with solutions } \gamma_1^* \quad \text{and} \quad \gamma_2^*. \quad \text{The maximum value of economic growth is:} \quad \gamma^*_1 = \frac{1}{2} \left[ -A + \sqrt{A^2 - 4B} \right]. \quad \text{To compute } \tau^*, \text{ note that in (12b):}
\[ \tau^* = 1 - \alpha + \alpha m \left( \frac{\rho}{\gamma} - \frac{S - 1}{S} \right) > 1 - \alpha \] for the transversality condition to be respected, which proves Proposition 3.

Appendix 4: The local stability of steady states

In the neighborhood of the Barro steady state, we have, with
\[ \psi(b_k) = g_k(b_k) + \delta - (g_k(b_k))^{1-\alpha} : \]
\[
J^B = \begin{bmatrix}
    c^B_k & c^B_i \\
    b^B_k & b^B_i
\end{bmatrix}
\begin{bmatrix}
    \alpha S (1-\tau)(1-\alpha) \left( \frac{dg_k}{db_k} \right)_i \left( (g_k^B)^{-\alpha} \right) + \frac{d\psi}{db_k} \\
    \gamma^B
\end{bmatrix}.
\]

Clearly \( \text{Det}(J^B) = -c^B_k \gamma^B < 0 \), so that the Barro steady state is a saddle path.

In the neighborhood of Solow steady state, we have:
\[
J^S = \begin{bmatrix}
    c^S_k & c^S_i \\
    b^S_k & b^S_i
\end{bmatrix}
\begin{bmatrix}
    \alpha S (1-\tau)(1-\alpha) \left( \frac{dg_k}{db_k} \right)_i \left( (g_k^S)^{-\alpha} \right) + \frac{d\psi}{db_k} \\
    \gamma^S
\end{bmatrix},
\]

Clearly \( \text{Det}(J^S) = -\alpha S (1-\tau)(1-\alpha) \left( \frac{dg_k}{db_k} \right)_i \left( (g_k^S)^{-\alpha} \right) > 0 \). To show that:
\[
\text{Tr}(J^S) = c^S_k + b^S_i \frac{d\psi}{db_k} > 0, \text{ note that: } b^S_k \frac{d\psi}{db_k} = \frac{rb_k (g_k)^{1-\alpha} \left[ (1-\alpha) - (g_k)^{1-\alpha} \right]}{\alpha g_k + (1-\alpha) \delta b_k}
\]
and
\[
c_k = (g_k)^{1-\alpha} - g_k - \delta = (1-\tau) (g_k)^{1-\alpha} - \delta + rb_k = r + (1-\tau)(g_k)^{1-\alpha} + rb_k, \text{ so:}
\]
\[
c_k + b^S_k \frac{d\psi}{db_k} = r + (1-\tau)(g_k)^{1-\alpha} + \frac{(1-\alpha)rb_k \left( (g_k)^{1-\alpha} + \delta g_k - g_k \right)}{\alpha g_k + (1-\alpha) \delta b_k} > 0
\]

Local stability of steady states if \( m > 0 \) (\( \lambda^j \) are the eigenvalues of \( J^j \))

<table>
<thead>
<tr>
<th>(%)</th>
<th>\text{Det}(J^B)</th>
<th>\text{Det}(J^S)</th>
<th>\text{Tr}(J^B)</th>
<th>\text{Tr}(J^S)</th>
<th>(\lambda^B_1, \lambda^B_2)</th>
<th>(\lambda^S_1, \lambda^S_2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>( m = 1 )</td>
<td>-1.48</td>
<td>1.83</td>
<td>10.03</td>
<td>27.47</td>
<td>18.17, -8.14</td>
<td>16.18, 11.29</td>
</tr>
<tr>
<td>( m = 2 )</td>
<td>-1.24</td>
<td>1.47</td>
<td>10.92</td>
<td>25.39</td>
<td>17.85, -6.93</td>
<td>16.41, 8.98</td>
</tr>
<tr>
<td>( m = 3 )</td>
<td>-0.91</td>
<td>1.04</td>
<td>12.29</td>
<td>22.83</td>
<td>17.51, -5.22</td>
<td>16.56, 6.27</td>
</tr>
</tbody>
</table>

For simulation values: \( \alpha = 0.6, \rho = 0.1, S = 2, \tau = 0.4, \delta = 0.05 \)
Appendix A: The model with public expenditure as a stock variable

Following the analysis of Barro & Sala-i-Martin (1992) and Futagami et al. (1993), we extend our model to public expenditure modeled as a stock variable. Replacing $g$ with $\dot{g} + \delta^g$ ($\delta^g$ is the public capital depreciation rate), the IS equilibrium (6) becomes: $\gamma_k = (g_k)^{1-a} - c_k - \delta - (\gamma_g + \delta^g)g_k$. The public debt evolution in (7) now is:

$$\gamma_b = \alpha(1-\tau)(g_k)^{1-a} - \delta + \frac{\tau(g_k)^{1-a} - (\gamma_g + \delta^g)g_k}{b_k},$$

with (5) unchanged. In steady state, public capital grows at the same rate as the other variables: $\gamma_g = \gamma$. Thus, (11) becomes:

$$H(g_k; \tau, m) = \gamma(g_k)[(m + \tau)g_k^{1-a} - g_k(\gamma(g_k) + \delta^g)] - mr(g_k)g_k^{1-a} = 0$$

with $r(g_k) = \alpha(1-\tau)(g_k)^{1-a} - \delta$ and $\gamma(g_k) = S[r(g_k) - \rho]$. We can easily remark that our steady state results are qualitatively unchanged (in particular Propositions 2 and 3). The two figures below represent respectively $F(g_k; \tau, m) = 0$ and.

For simulation values: $\alpha = 0.6, \rho = 0.05, S = 1, \tau = 0.4, \delta = 0.05$

Note that $\gamma^2$ function becomes (we extend the more general expression (10c), and we express the inverse function $g_k^\alpha(\gamma^2)$ for simplicity): $g_k^\alpha(\gamma^2) = \frac{\gamma^2\left[ m\left(\frac{S-1}{S}\right) + \tau \right] - \rho m}{\gamma^2\left[ 1 - \mu\left(\frac{S-1}{S}\right) \right] + \rho \mu \gamma^2 + \delta^g}$.

This function describes a growing relation between $\gamma^2$ and $g_k$ for $\gamma^2 < \bar{\gamma}$, and a decreasing curve for $\gamma^2 > \bar{\gamma}$. $\bar{\gamma}$ is the only positive rate of growth that maximizes $g_k^\alpha(\gamma^2)$. For instance, if $\mu = \delta^g = 0, \bar{\gamma} = 2\rho m\left[ m\left(\frac{1-S}{S}\right) + \tau \right]^{-1}$. The $\gamma^2$ curve (12a) is unchanged. It is easy to
compute: $\frac{dg_k}{dm} \bigg|_{\gamma^* \text{ given}} < 0$ and $\frac{dg_k}{d\mu} \bigg|_{\gamma^* \text{ given}} < 0$, which extends Proposition 2 to a model with public investment expenditures (the $\gamma^*$ curve moves towards the left).

To extend Proposition 3, note that $\frac{d\gamma}{d\tau} = -\frac{\partial H}{\partial \tau} \bigg/ \frac{\partial H}{\partial \gamma} = 0$ if

$$(\gamma + \delta^s) \left( g_k (\tau^*) \right)^{\alpha} = \frac{(1-\tau^*)(1-\alpha)}{\alpha}.$$  

Now, in the government budget constraint:

$$(\gamma + \delta^s) g_k^\alpha = \left( \tau - m \left( \frac{r-\gamma}{\gamma} \right) \right).$$

Thus, the tax rate $\tau^*$ that maximizes economic growth is still:

$$\tau^* = 1 - \alpha + \alpha m \left( \frac{\rho}{\gamma} - \frac{S-1}{S} \right) > 1 - \alpha.$$  

As regards transitional dynamics, in an appendix available on request, we show that the two steady states have the same local stability properties. Note that, with public investment, system (10) becomes a three variables system, with an additional accumulation (predetermined) variable $g$, which does not allow for two-dimensional representations. Using simulations, we show that, except for a scale effect, none of our results are affected: the low equilibrium is unstable, with a negative and two positive eigenvalues, while the high equilibrium is a saddle path, with a positive and two negative eigenvalues.