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Abstract

By defining the Variable Output Elasticities Cobb-Douglas function, this article shows that a large class of production functions can be approximated by a Cobb-Douglas function with non-constant output elasticity. Compared to standard flexible functions such as the Translog function, this framework has several advantages. It requires only the use of the first order approximation while respecting the theoretical curvature conditions of the isoquants. This greatly facilitates the deduction of linear input demands function without the need of involving the duality theorem. Moreover, it allows for a generalization of the CES function to the case where the elasticity of substitution between each pair of inputs is not necessarily the same.

Keywords: flexible production functions, Cobb-Douglas function, CES function.

JEL-Code: D24, E23.

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1. Introduction

In their influential contribution to economic theory, Cobb and Douglas (1928) introduced a class of production functions that was named after them. Since, the Cobb-Douglas (CD) function has been (and is still) abundantly used by economists because it has the advantage of algebraic tractability and of providing a fairly good approximation of the production process. Its main limitation is to impose an arbitrary level for substitution possibilities between inputs. To overcome this weakness, important efforts have been made to develop more general classes of production function with as a corollary a strong increase in complexity (for a recent survey see e.g. Mishra, 2010).

Arrow et al. (1961) introduced the Constant Elasticity of Substitution (CES) production function which has the advantage to be a generalization of the three main functions that were used previously: the linear function (for perfect substitutes), the Leontief function (for perfect complements) and the CD function, which assume respectively an infinite, a zero and a unit elasticity of substitution (ES) between production factors.

A limitation of the CES function is known as the impossibility theorem of Uzawa (1962) - McFadden (1963) according to which the generalization of the class of function proposed by Arrow et al. (1961) to more than two factors imposes a common ES between factors. To allow for different degrees of substitutability between inputs, Sato (1967) proposed the approach of nested CES functions which has proved very successful in general equilibrium modeling and econometric studies because of its algebraic tractability. Although this method is flexible, some substitution mechanisms remain constrained[1] and the choice of the nest structure is often arbitrary.

To overcome this limit, several "flexible" production functions have been proposed such as the Generalized Leontief (GL) (Diewert, 1971) and Transcendental Logarithmic (Translog) function (Christensen et al., 1973). These are second order approximations of any arbitrary twice differentiable production functions (for a formal proof in the case of the Translog function see e.g. Grant, 1993). They have the advantage not to impose any constraint on the value of the ES between different pairs of inputs but their use is much more complex. This at least partly explains their little

[1] To see this, consider the following nested CES function: $Q = C^{ES}\left(C^{ES}(I_1^{nput}, I_2^{nput}), I_3^{nput}\right)$. Possibilities of substitution between input 1 and 2 may differ from the ES between 1 and 3, but those between 1 and 3 are identical to those between 2 and 3.

success in general equilibrium modeling compared to the nested CES approach[2]. Two difficulties are particularly limiting:

- Due to the complexity of the function, the demands for inputs cannot be derived directly from the specification of the flexible production function. Using the Sheppard lemma and the duality theorem, the demands for inputs are derived from a second order approximation of the cost function at the optimum. This approach raises two issues. First, one needs to have data about costs in order to derive their relation with prices and production over time. Second, the presence of rigidity in inputs (in particular in equipment) do not guaranty that the approximation is at the optimum.
- Because of the use of a linear approximation, it is often difficult to impose the theoretical curvature conditions of the isoquants (see Diewert and Wales, 1987). This may generate poor results in the case of important variations of prices. As a consequence, the approach may be unsuitable for use in applied general equilibrium modeling because it may lead to the failure of the algorithm[3].

Whereas the existing literature has attempted to overcome the weakness of the CD function by proposing more general but also complex alternatives, we remain here in the tractable framework of the CD function and investigate the condition under which it can be used to approximate more flexible functions. We show that any homogeneous production function can be approximated by a CD function where the output elasticities are not constant (unless the ES between inputs is equal to one). As we shall see, this approach has several advantages:

- It requires only the use of the first order approximation which avoid the tedious algebraic of the second order approximation used in flexible function. Compared to the approximation used in flexible function, the point of approximation is not fixed but varies when the quantity of each input varies. This allows for a well-behaved production function that respects the theoretical curvature conditions of the isoquants.
- This linear approximation allows for the derivation of algebraically tractable input demand functions without involving the duality theorem and the approximation of the cost function at the optimum.

[2] See Jorgenson (1998) for the use of Translog function in general equilibrium modeling.

[3] For a discussion see Perroni and Rutherford (1995) who argue that traditional flexible functional forms suffer from an excess of flexibility. They advocate for the use of the nested CES cost function which is globally well-behaved and can provide a local approximation to any globally well-behaved cost function.

- This greatly facilitates the deduction of linear input demands that can be estimated using standard linear regression models.
- This new class of function allows for a generalization of the CES to the case where the ES between each pair of inputs are not necessarily the same and hence for avoiding the limitation of the impossibility theorem and the use of the nested CES approach.
- This allows for easily introducing different levels of elasticity between production factors. In particular, changing the level of elasticity between factors is easier than in the nested CES approach since it does not require changing the structure of the nest.

Section 2 defines the Variable Output Elasticities CD (VOE-CD) function in the general case of J inputs and shows that the CD function can be seen as an approximation of any homogeneous function. Section 3 draws the implication of the impossibility theorem in the case of three or more inputs. Section 4 derives the demand for inputs in the case of a VOE-CD production function. Section 5 compares numerically the CES function with its approximation by a VOE-CD function in the case of two inputs. Section 6 concludes.

2. The variable output elasticities Cobb Douglas function

Let us assume that the technology of production can be represented by a continuous and twice differentiable function, homogeneous of degree 1 (constant returns to scale), increasing ($Q'(X_j) > 0$) and strictly concave ($Q''(X_j) < 0$) reflecting the law of diminishing marginal returns[4]:

$$Q = Q(X_j) \tag{1}$$

Where X_j is the quantity of input (or production factor) $j = [1; J]$ used to produce the quantity of production (or output) Q[5]. The first order Taylor approximation of this function is:

[4] In this paper, lower-case variables are in logarithm: $x = \ln(X)$. The first and second derivatives of a function Q with respect to X are respectively $Q'(X) = \dfrac{\partial Q(X)}{\partial X}$ and $Q''(X) = \dfrac{\partial^2 Q(X)}{(\partial X)^2}$. Variables in growth rate are referred to as $\dot{X} = x'(X) = \partial X / X$. All parameters written in Greek letter are positive.

$$\partial Q = \sum_{j=1}^{J} Q'(X_j)\partial X_j \qquad (2)$$

In the standard approach used for the construction of a flexible function such as the GL or Translog functions, Equation (2) is expanded and includes also the elements of the second order approximation. Moreover, the first and second partial derivatives of the production function are assumed constant and equal to their value at the point of approximation. In the Translog function, the first order polynomial is simply the CD function. The second order polynomial provides a correction that allows for approximating other types of production function around the point of reference by making the isoquant more (or less) convex. Since Equation (2) is not a log transformation, fixing the marginal productivity leads to the linear production function that characterizes perfect substitutes. One could include the second order polynomial to make the isoquant more convex in order to approximate production functions with lower ES. One limit of this approach is that this correction is only valid around the reference point. The further from this reference point, the poorer the approximation. Moreover, when the distance from the reference point becomes too large, the standard theoretical properties of the production function (strictly increasing and concave) are not even satisfied anymore[6]. Using a higher order of approximation could reduce this problem but it would not rule it out and additionally it would lead to a huge increase in complexity.

Here we propose another approach. We only consider the first order expansion (2) but we do not assume a constant marginal productivity evaluated at a reference point. On the contrary, we provide a specification to the marginal productivity that is consistent with two standard results of economic theory: [1] from the definition of the ES, the marginal productivity of a given input depends on the quantity of this input used relatively to the other inputs; [2] from optimality, the marginal productivity of a given input varies with its price.

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[5] If one assumes that X_j is the quantity of "efficient" input and $Q = Y^{1/\theta}$ where Y is the level of production and θ the level of returns to scale, all the results presented below can be generalized to account for returns to scale and technical progress.

[6] This problem can easily be illustrated. For instance, assume that the production function is approximated by a second order Taylor polynomial around zero and that only one input varies: $Q(X) = \alpha_0 X - \alpha_1 X^2 / 2$ with $\alpha_0 > 0$. The property of strict concavity ($Q''(X_j) < 0$) requires $\alpha_1 > 0$. But then the other fundamental property of a strictly increasing production function ($Q'(X_j) > 0$) is satisfied only for $X < \alpha_0 / \alpha_1$.

Assuming production and the other inputs constant, one can notice that Equation (2) provides the textbook specification of the Marginal Rate of Substitution (MRS) between each pair of inputs. The latter is equal to the ratio between the marginal productivities of each input:

$$MRS_{jj'} = \frac{\partial X_j}{\partial X_{j'}} = -\frac{Q'(X_{j'})}{Q'(X_j)} \tag{3}$$

The MRS being the first derivative of the isoquant (the slope of the iso-production curve), its integral is the isoquant itself. We can use this property to approximate various class of production functions by formulating hypothesis about the specification of the marginal productivity of each input. For instance, in the case of perfect substitutes, the MRS is constant and the isoquant is a straight line. For less substitutable input, the MRS is increasing and the isoquant is more convex. Assuming a single reference point where the combination for the levels of production and inputs is known, the integral of the MRS from this point allows for drawing any isoquant and thus for deriving any production function. To do so, let us conveniently rewrite (2) in growth rate:

$$\dot{Q} = \sum_{j=1}^{J} \frac{Q'(X_j)X_j}{Q(X_j)} \dot{X}_j \tag{4}$$

The Euler's Theorem states that a function which is homogeneous of degree 1 can be express as the sum of its arguments weighted by their first partial derivatives:

$$Q(X_j) = \sum_{j=1}^{J} Q'(X_j)X_j \tag{5}$$

Incorporating (5) into (4), we see that the first order approximation (2) can equivalently be written as:

$$\dot{Q} = \sum_{j=1}^{J} \varphi_j \dot{X}_j \Leftrightarrow \partial q = \sum_{j=1}^{J} \varphi_j \partial x_j \tag{6}$$

with

$$\varphi_j = \frac{Q'(X_j)X_j}{\sum_{j'=1}^{J} Q'(X_{j'})X_{j'}} = \left(\sum_{j'=1}^{J} \frac{Q'(X_{j'})X_{j'}}{Q'(X_j)X_j} \right)^{-1} \tag{7}$$

Where φ_j is the output elasticity (OE) of input *j*.

Because of the hypothesis of constant returns to scale, $\sum_{j=1}^{J}\varphi_j = 1$ and Equation (6) is nothing else but a CD function written in growth rate (or logarithmic first difference) and homogeneous of degree 1. However here the OEs, φ_j, are not necessarily constant as in the standard CD function written in level. For this reason and in order to avoid any ambiguity, we shall from now on define Equations (6) and (7) as the Variable Output Elasticities Cobb-Douglas (VOE-CD) function whereas we shall define the standard CD function as the Constant Output Elasticities Cobb-Douglas (COE-CD) function.

It is important to emphasize that the VOE-CD function (6) is written in logarithmic first difference. This is an important difference with the COE-CD function written in level:

$$Q = \prod_{j=1}^{J} X_j^{\varphi_j} \Leftrightarrow q = \sum_{j=1}^{J}\varphi_j x_j \qquad (8)$$

When the OEs are constant, the specification in level (8) and in logarithmic first difference (6) are perfectly equivalent. This is not the case when the OEs vary. As we shall see in Section 5, the specification in level (8) with non constant OEs (7) provides a poor approximation of the specification in logarithmic first difference (6) in the case of an important change in the ratio between marginal productivities (i.e. between input prices).

Although we have shown that any function (1) can be approximated by a VOE-CD function (6)-(7), the particular cases embodied in the CES function are worth few developments (Section 5 compares graphically the CES function and its VOE-CD approximation in the case of 2 inputs). Expectedly, the COE-CD function is one particular case of the VOE-CD function. It corresponds to the case where the ES between each input is equal to one. This is easy to see when one introduces the definition of the ES proposed by Hicks (1932) and Robinson (1933) between inputs j and j' ($\eta_{jj'}$). This elasticity measures the change in the ratio between two factors of production due to a change in their relative marginal productivity, i.e. in the MRS:

$$\begin{aligned} -\eta_{jj'} &= \frac{\partial \ln(X_j / X_{j'})}{\partial \ln\left(Q'(X_j)/Q'(X_{j'})\right)} \Leftrightarrow \frac{Q'(X_j)}{Q'(X_{j'})} = \xi_{jj'}\left(\frac{X_j}{X_{j'}}\right)^{-1/\eta_{jj'}} \\ &\Leftrightarrow \dot{X}_j - \dot{X}_{j'} = -\eta_{jj'}\left(\dot{Q}'(X_j)/\dot{Q}'(X_{j'})\right) \end{aligned} \qquad (9)$$

Where $\xi_{jj'} = \dfrac{\tilde{\varphi}_j}{\tilde{\varphi}_{j'}}$ is a constant of integration that reflects the relative weight of each input in the production function: $\sum_{j=1}^{J} \tilde{\varphi}_j = 1$[7]. Notice that this definition of the ES is symmetric: $\eta_{jj'} = \eta_{j'j}$. Extrapolating the results presented below to the asymmetric case using the definition of the ES proposed by Morishima and advocated by Blackorby and Russell (1989) is straightforward but complicates their algebraic exposition[8].

Integrating (9) into (7), the OE becomes a function of the ratios between input:

$$\varphi_j = \left(\sum_{j'=1}^{J} \xi_{j'j} \left(\frac{X_{j'}}{X_j} \right)^{1-1/\eta_{jj'}} \right)^{-1} \tag{10}$$

In the case of a CEO-CD function, all the ES are equal to one ($\eta_{jj'} = 1$) and any change in the ratio between the two inputs is exactly compensated by the change in their relative marginal productivity. As a consequences, the OEs are constant: $\varphi_j = \left(\sum_{j'=1}^{J} \xi_{j'j} \right)^{-1} = \dfrac{\tilde{\varphi}_j}{\sum_{j'=1}^{J} \tilde{\varphi}_{j'}}$.

In the case of a Leontief function, all the ES are equal to zero ($\eta_{jj'} = 0$). The OE of input *j* tends toward zero (resp. one) if the input ratio $\dfrac{X_j}{X_{j'}}$ increases (resp. decreases). This result reflects the perfect complementary between inputs: increasing the quantity of input *j* while leaving the quantity of the other inputs constant does not increase the level of production because the marginal productivity falls to zero (thus $\varphi_j \to 0$); increasing the quantity of the other inputs *j′* while leaving the quantity of the input *j* constant does not increase the level of production either but increases the marginal productivity of input *j* (thus $\varphi_j \to 1$).

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7 In the general equilibrium models, these weights are calibrated at a reference point in time using base year data.

8 This generalization would require to change Equation (9) into $\dot{X}_j - \dot{X}_{j'} = -\eta_{jj'} \dot{Q}'(X_j) + \eta_{j'j} \dot{Q}'(X_{j'})$ with $\eta_{jj'} \neq \eta_{j'j}$.

In the case of perfect substitutes, all the ES tends toward infinity ($\eta_{jj'} \to +\infty$) and the OE tends toward $\varphi_j = \frac{\tilde{\varphi}_j X_j}{\sum_{j'=1}^{J} \tilde{\varphi}_{j'} X_{j'}}$. In this case, Equation (6) tends towards the linear production function that characterizes perfect substitutes, $\partial Q = \sum_{j=1}^{J} \tilde{\varphi}_j \partial X_j \Leftrightarrow Q = \sum_{j=1}^{J} \tilde{\varphi}_j X_j$, where the marginal productivity of each input is always constant and equal to $\tilde{\varphi}_j$ whatever the level of the ratio between inputs is.

3. The impossibility theorem

Because of the impossibility theorem, the case of a common ES between all factors is the only possible case where the ES are constant between every pair of inputs. When the ES differs between pairs, at least one ES is not constant. The reason is that the system of Equations (9) is over-identified for a number of inputs higher that 2 ($J > 2$). For instance, in the case of three inputs, there are three possible relations:

$$\dot{X}_1 - \dot{X}_2 = -\eta_{12} \left(\dot{Q}'(X_1) - \dot{Q}'(X_2) \right) \tag{11}$$

$$\dot{X}_1 - \dot{X}_3 = -\eta_{13} \left(\dot{Q}'(X_1) - \dot{Q}'(X_3) \right) \tag{12}$$

$$\dot{X}_2 - \dot{X}_3 = -\eta_{23} \left(\dot{Q}'(X_2) - \dot{Q}'(X_3) \right) \tag{13}$$

Subtracting (11) to (12) defines unambiguously the ratio between Input 2 and Input 3. Equation (13) is thus redundant and the ES between 2 and 3 has to be:

$$\eta_{23} = \frac{\eta_{12} \dot{Q}'(X_2) - \eta_{13} \dot{Q}'(X_3) + (\eta_{13} - \eta_{12}) \dot{Q}'(X_1)}{\dot{Q}'(X_2) - \dot{Q}'(X_3)} \tag{14}$$

If the ES between Input 1 and Input 2 equals the one between 1 and 3 ($\eta_{12} = \eta_{13}$), it also equals the one between 2 and 3 ($\eta_{12} = \eta_{23}$) and all ES are constant and identical. But if the ES between Input 1 and Input 2 and between 1 and 3 are different ($\eta_{12} \neq \eta_{13}$), the ES between Input 2 and Input 3 is not constant and depends on the ES between 1 and 2 and between 1 and 3 [see Equation (14)]. The ES between 1 and 2 can take an infinite number of values depending if the variation of marginal productivity comes from Input 2 or Input 3. Moreover, the ratio between the marginal productivities of Input 2 and Input 3 that enters in the OE Equation (7) is pre-defined:

$$\dot{Q}'(X_2) - \dot{Q}'(X_3) = -\frac{1}{\eta_{12}}\dot{X}_2 + \frac{1}{\eta_{13}}\dot{X}_3 + \left(\frac{1}{\eta_{12}} - \frac{1}{\eta_{13}}\right)\dot{X}_1 \tag{15}$$

We will see in the next section that the impossibility theorem has important implications on the specification of the input demand function.

4. The demand for inputs

We now deduce the demand for inputs in the case of the VOE-CD production function (6) and (7). Driven by a maximizing profit behavior, the producer chooses her demand for each input by minimizing her production cost (16) subject to the technical constraint (1):

$$C = \sum_{j=1}^{J} P_j^X X_j \tag{16}$$

Where P_j^X is the price of input *j*. The Lagrangian to this problem is:

$$L = C - \lambda\left(Q - Q(X_j)\right) \tag{17}$$

The well-known first order necessary condition ($L'(X_j) = 0$) says that at the optimum, the ratio between the marginal productivities of two inputs equals the ratio between their prices[9]:

$$Q'(X_j) / Q'(X_{j'}) = P_j^X / P_{j'}^X \tag{18}$$

The combination of Equations (7) and (18) shows that, at the optimum, the OE of Input *j* in the VOE-CD function corresponds to the cost share of input *j*:

[9] The first order conditions is sufficient for optimality because of the assumption of a strictly concave production function (1).

$$\varphi_j = \frac{P_j^X X_j}{\sum_{j'}^{J} P_{j'}^X X_{j'}} \tag{19}$$

Under the assumption that the sales' revenues of production are totally exhausted by the remuneration of the factors of production, Equation (19) is also the share (in value) of input *j* in the production and allows for calibrating the VOE-CD function (6) at a base year in the exact same way it is generally used to calibrate a COE-CD function. This standard calibration procedure is at the origin of the controversy about the robustness of the empirical success of the COE-CD function. According to Samuelson (1979), the CD econometric estimation would do nothing more than reproducing the income distribution identity (see also Felipe and Adams, 2005). As previously mentioned (Section 2), the OEs (19) are constant only in the case of a unit ES (i.e. the case of a COE-CD function) since any variation in the input price ratio is exactly compensated by a change in the corresponding input ratio. The OEs may however be constant in the long run for any configuration of ES provided that two hypotheses are satisfied: [1] the stability of the ratios between input prices; [2] every input grows at the same rate as production. These are the standard hypotheses used to define the stationary state of an economy in growth theory and macroeconomic textbook (e.g. Romer, 1996). They are roughly verified in the long run since the labor and capital shares are generally relatively stable over time. This suggests that the COE-CD function provides a good approximation for the analysis of long term macroeconomic phenomena and supports its theoretical and empirical success in many studies on long term economic growth. Our analysis gives thus a more favorable view of the CD empirical success than the controversial views of Samuelson (1979) and others.

Combining the first order conditions (18) to the definition of the ES (9) and the production function (6) gives the demand for each factor as a positive function of output and a negative function of the relative prices between production factors:

$$\dot{X}_j = \dot{Q} - \sum_{\substack{j'=1 \\ j' \neq j}}^{J} \eta_{jj'} \varphi_{j'} (\dot{P}_j^X - \dot{P}_{j'}^X) \tag{20}$$

As already mentioned, because of the impossibility theorem, certain ES are constrained in the case of more than 2 inputs. The demand for only one input (*j*) can be defined as Equation (20) whereas the demand for the other inputs is defined relatively to Input *j* according to the definition of the ES (9):

$$\dot{X}_{j'} - \dot{X}_j = -\eta_{jj'}(\dot{P}^X_{j'} - \dot{P}^X_j) \tag{21}$$

Assuming a constant ES between inputs, $\eta_{jj'} = \eta$, the demand for production factors (20) expectedly simplifies to the specification that is derived from a CES function. The input demand depends only on the relative price between the input price and the production price, P^Q (under the assumption of profit exhaustion):

$$\dot{X}_j = \dot{Q} - \eta(\dot{P}^X_j - \dot{P}^Q) \tag{22}$$

with
$$\dot{P}^Q = \sum_{j=1}^{J} \varphi_j \dot{P}^X_j \tag{23}$$

One may notice that the above specification is similar to the consumer's demand for goods derived from a CES utility function. Here the price index (P^Q) is nothing else but the linear approximation of the Dixit and Stiglitz (1977) CES price index (e.g. Blanchard and Kiyotaki, 1987).

5. Transforming a CES into a Cobb-Douglas function: the case of 2 inputs

In Section 2, we have shown analytically that any homogeneous function can be approximated by a VOE-CD function. As a numerical illustration, we now compare the CES function and its approximation with a VOE-CD function. We consider the case of two inputs that allows for graphical representation. Assuming that the ES between Input 1 and Input 2 is η, the specification of the CES function is:

$$Q = \left(\tilde{\varphi}_1 X_1^{(\eta-1)/\eta} + \tilde{\varphi}_2 X_2^{(\eta-1)/\eta}\right)^{\eta/(\eta-1)} \tag{24}$$

The corresponding MRS calculated according to (3) is:

$$MRS_{1,2} = \frac{\partial X_2}{\partial X_1} = -\frac{Q'(X_1)}{Q'(X_2)} = -\frac{\tilde{\varphi}_1}{\tilde{\varphi}_2}\left(\frac{X_1}{X_2}\right)^{-1/\eta} \tag{25}$$

From Equations (6), (7) and (9), the VOE-CD approximation of the CES function (24) is:

$$\dot{Q} = \varphi_1 \dot{X}_1 + \varphi_2 \dot{X}_2 \Leftrightarrow \partial q = \varphi_1 \partial x_1 + \varphi_2 \partial x_2 \qquad (26)$$

with $$\varphi_1 = \frac{\tilde{\varphi}_1 X_1^{1-1/\eta}}{\tilde{\varphi}_1 X_1^{1-1/\eta} + \tilde{\varphi}_2 X_2^{1-1/\eta}} \quad \text{and} \quad \varphi_2 = 1 - \varphi_1 \qquad (27)$$

Section 2 has shown in the general case of *J* inputs that the MRS can be derived from the VOE-CD function assuming that production is constant. The reformulation (26) and (27) assuming a constant production ($\dot{Q} = 0$) leads thus to the MRS (25) calculated directly from the CES function (24). By construction, the VOE-CD function has in addition the advantage of well-behaved isoquants, i.e. strictly decreasing ($MRS_{1,2} < 0$) and strictly convex ($\frac{\partial MRS_{1,2}}{\partial \left(X_1 / X_2 \right)} > 0$).

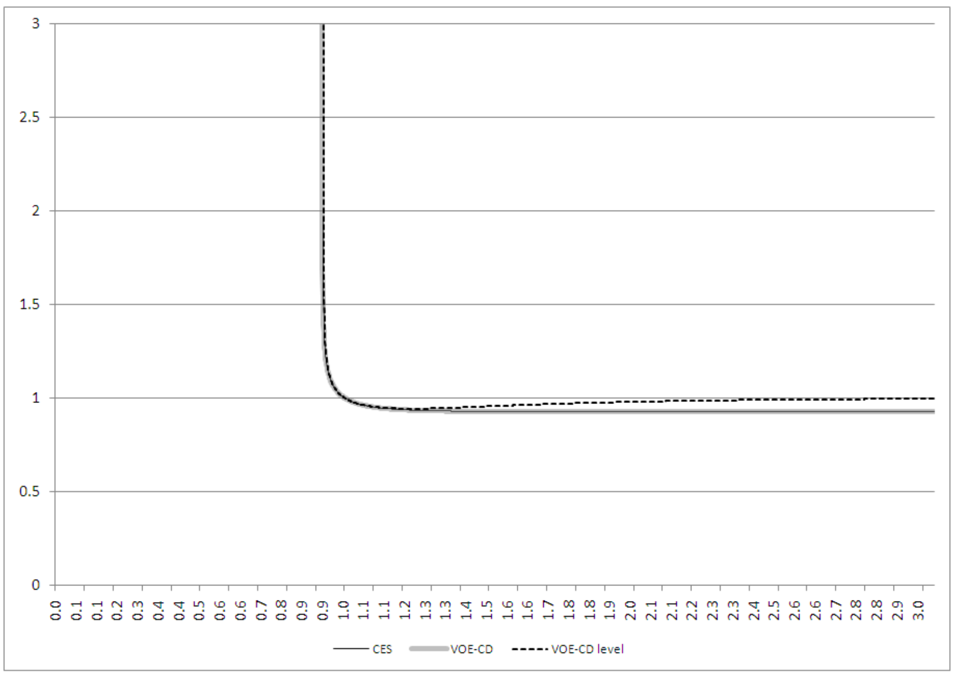
Taking the integral of Equation (25) from a reference point $(Q_0; X_{1,0}; X_{2,0})$, where the combination of production and inputs levels is known (e.g. the combination at the base year), allows for drawing the isoquant:

$$X_{2,t} - X_{2,0} = \int_0^t \partial X_2 = \int_0^t -\frac{\tilde{\varphi}_1}{\tilde{\varphi}_2} \left(\frac{X_1}{X_2} \right)^{-1/\eta} \partial X_1 \qquad (28)$$

In numerical simulation, the precision of the approximation provided by Equation (28) depends on the integral step used: if the step is very small, i.e. if $\partial X_1 \to 0$, the approximation converges to the exact specification (24). The higher the step, the less convex the approximated isoquant compared to the exact one. This means that the approximation of the CES function with a VOE-CD function tends to exaggerate the actual level of substitution between inputs. This means also that for a given step, the quality of the approximation is better for high levels of ES than for low levels. In the limit case of perfectly substitutable inputs ($\eta \to \infty$), the approximation is good even for a high step since in this case the MRS is constant and independent on the ratio between inputs: $MRS_{1,2} \to -\frac{\tilde{\varphi}_1}{\tilde{\varphi}_2}$.

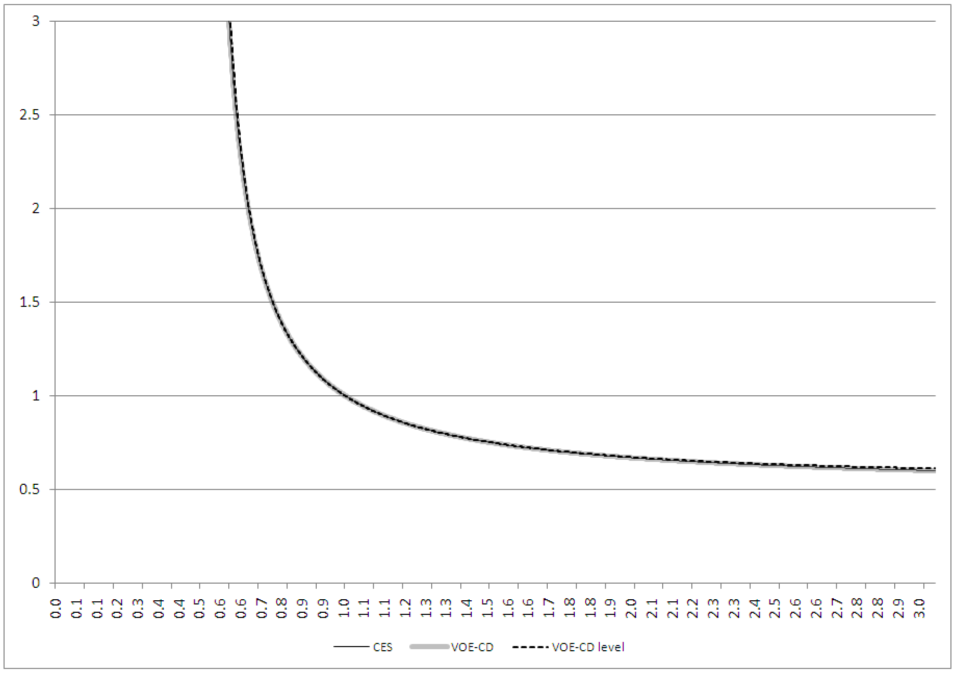
Figure 1 to Figure 5 compares the isoquants calculated with a CES function (24) (referred as CES) and its VOE-CD calculated according to (28) (referred as VOE-CD) under the following hypotheses: $Q = 1$ and $\tilde{\varphi}_1 = \tilde{\varphi}_2 = 0.5$. Because the step chosen is small ($\partial X_1 = 0.001$), there is no visual difference between the two curves. The comparison of curves using different steps is shown in Figure 6 and Figure 7. We see that using a step that is ten times higher ($\partial X_1 = 0.01$) still provides a good approximation with hardly no visual difference compared to the exact isoquant. This is not the case for a step that is 100 or 200 times higher ($\partial X_1 = 0.1$ or 0.2) and for which the isoquant appears clearly less convex. As expected from the above discussion, the bias is more important when inputs are more complement (Figure 6: ES = 0.5) than substitute (Figure 7: ES = 5).

Figure 1. Isoquant with ES = 0.1



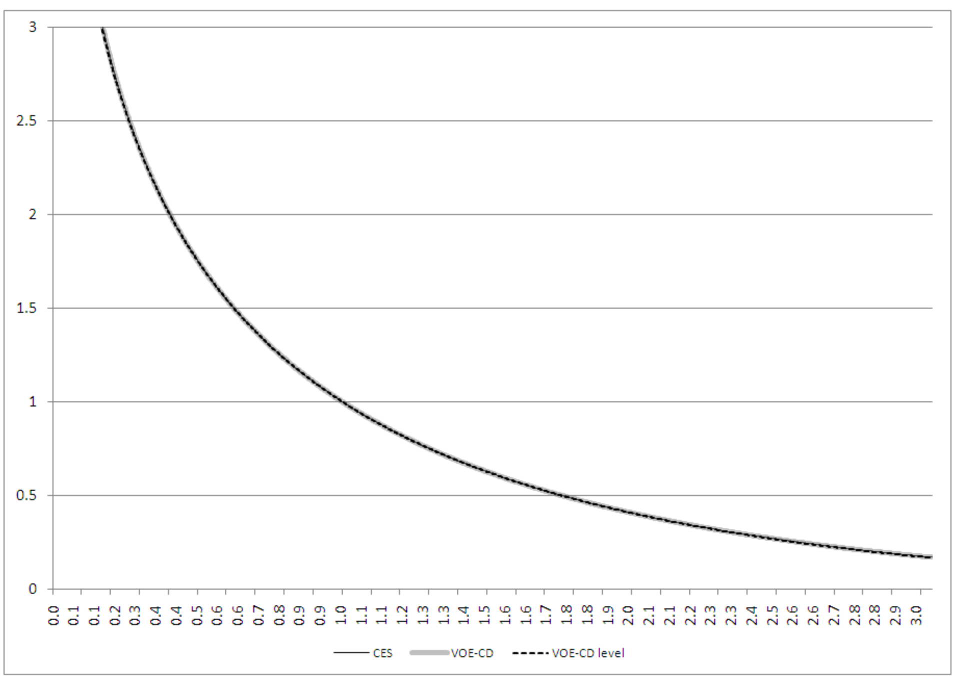
Key: $Q = 1$; $\tilde{\varphi}_1 = \tilde{\varphi}_2 = 0.5$; X-axis: Input 1 ($X_1$); Y-axis: Input 2 ($X_2$); CES: isoquant of the CES function [Equation (24)]; VOE-CD: isoquant of the VOE-CD function [Equations (26) and (27)] with a step $\partial X_1 = 0.001$; VOE-CD level: isoquant of the VOE-CD function written in level [Equations (29) and (27)]; Source: author's calculation.

Figure 2. Isoquant with ES = 0.5



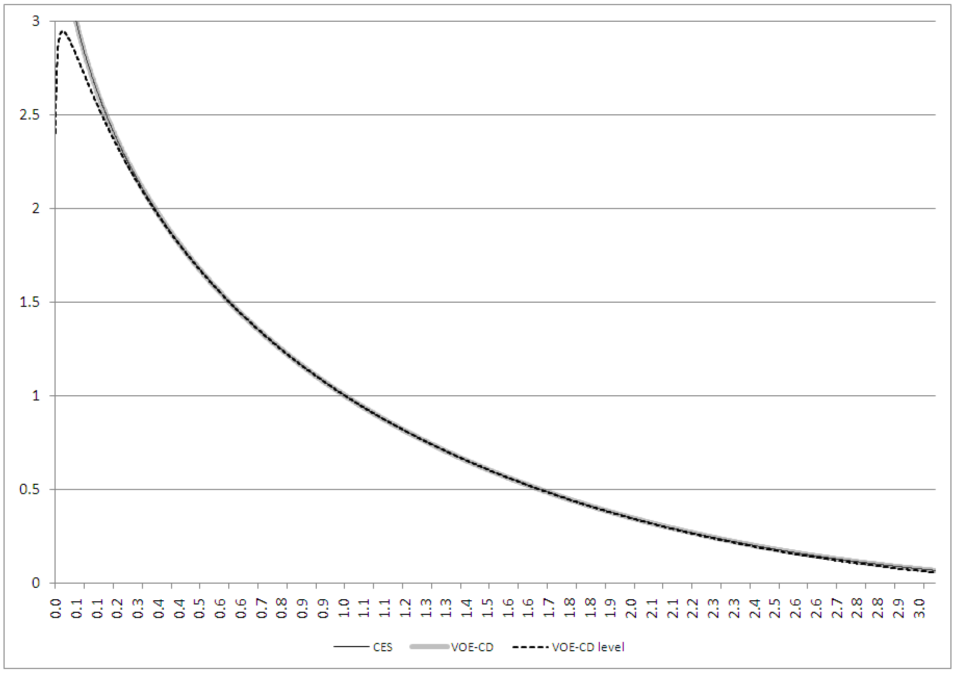
Key: idem Figure 1.

Figure 3. Isoquant with ES = 1.5



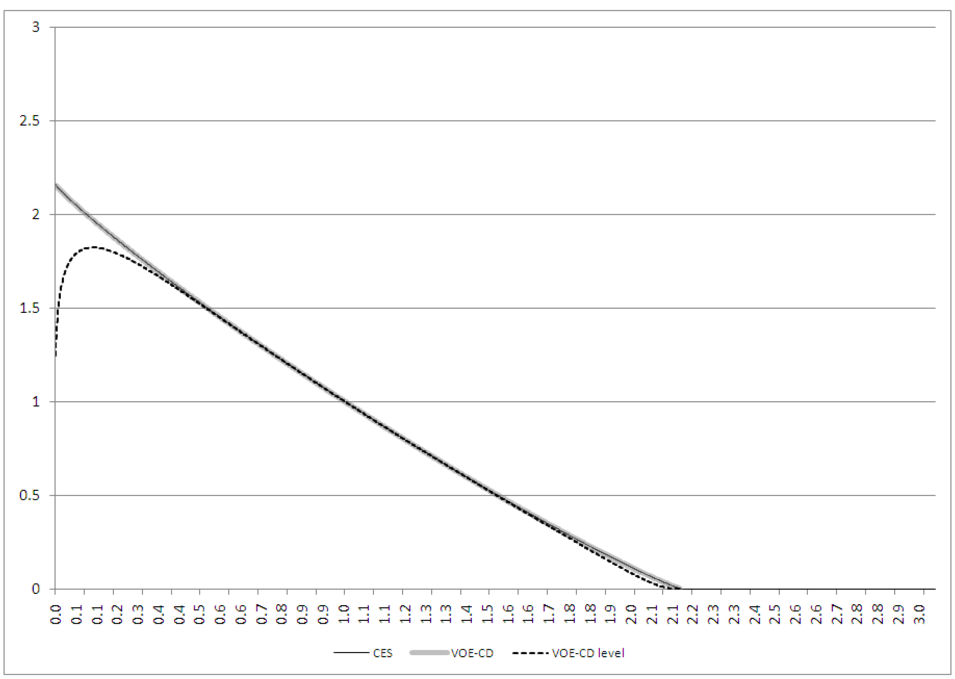
Key: idem Figure 1.

Figure 4. Isoquant with ES = 2



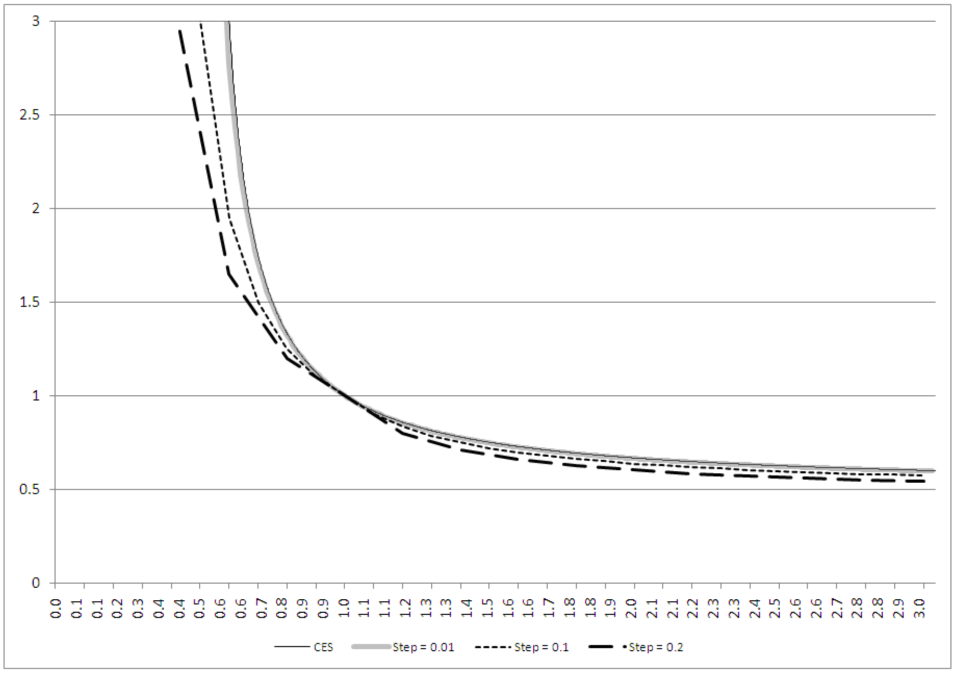
Key: idem Figure 1.

Figure 5. Isoquant with ES = 10



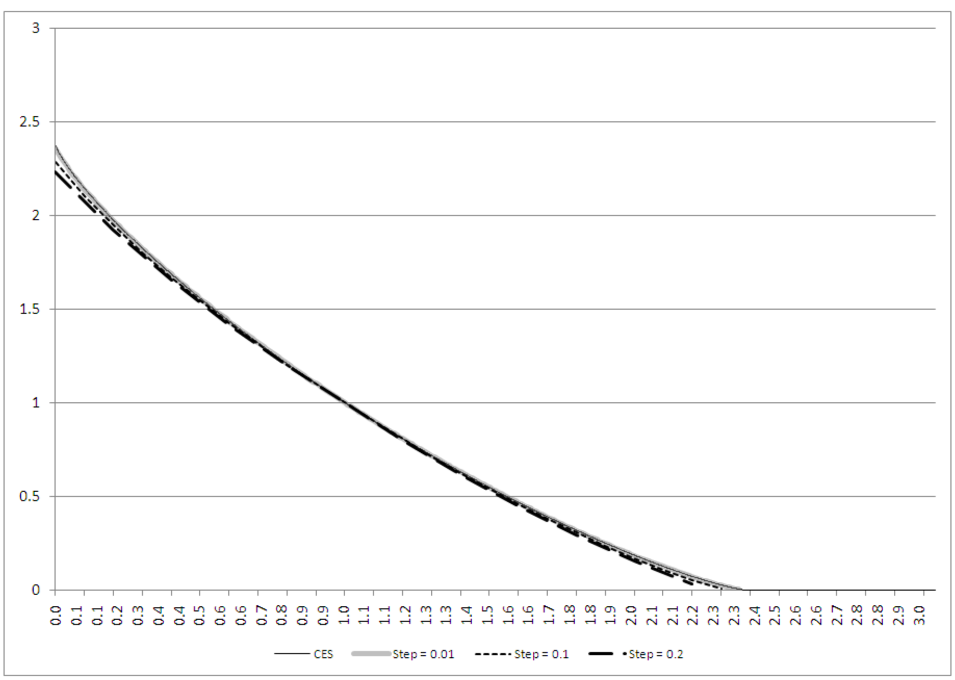
Key: idem Figure 1.

Figure 6. Isoquant with ES = 0.5 for various steps



Key: $Q=1$; $\tilde{\varphi}_1=\tilde{\varphi}_2=0.5$; X-axis: Input 1 ($X_1$); Y-axis: Input 2 ($X_2$); CES: isoquant of the CES function [Equation (24)]; Step = 0.01 (resp. 0.1; 0.2): isoquant of the VOE-CD function [Equations (26) and (27)] with a step $\partial X_1=0.01$ (resp. 0.1; 0.2); Source: author's calculation.

Figure 7. Isoquant with ES = 5 for various steps



Key: idem Figure 6.

In Section 2, we have emphasized that, for accuracy, the VOE-CD function should be written logarithmic first difference (and not in level as for the COE-CD function). Still, it is interesting to investigate the behavior of a VOE-CD function in level in order to measure the importance of its bias. To do so, we have drawn an additional curve in Figure 1 to Figure 5 (referred as VOE-CD level) which is a reformulation of the VOE-CD function (26) in level:

$$q = \varphi_1 x_1 + \varphi_2 x_2 \Leftrightarrow Q = X_1^{\varphi_1} X_2^{\varphi_2} \tag{29}$$

The specification for the OEs remains the one of Equation (27). This specification in level is equivalent to the specification in first difference (26) only if the OEs are constant, i.e. in the case of a unit ES between inputs. Indeed in such a case, Equation (29) is the integral of Equation (26). For any other level of ES, Equation (29) may not provide a good approximation especially for level of ES far from one and/or for important changes in the input ratio. Figure 1 to Figure 5 confirm this expectation by showing that the curvature conditions are only satisfied locally around a reference point where the input ratio is equal to one[10]. At this point, the MRS is equal to the relative weight between each input, $MRS_{1,2} = -\frac{\tilde{\varphi}_1}{\tilde{\varphi}_2}$, irrespective of the chosen level of ES.

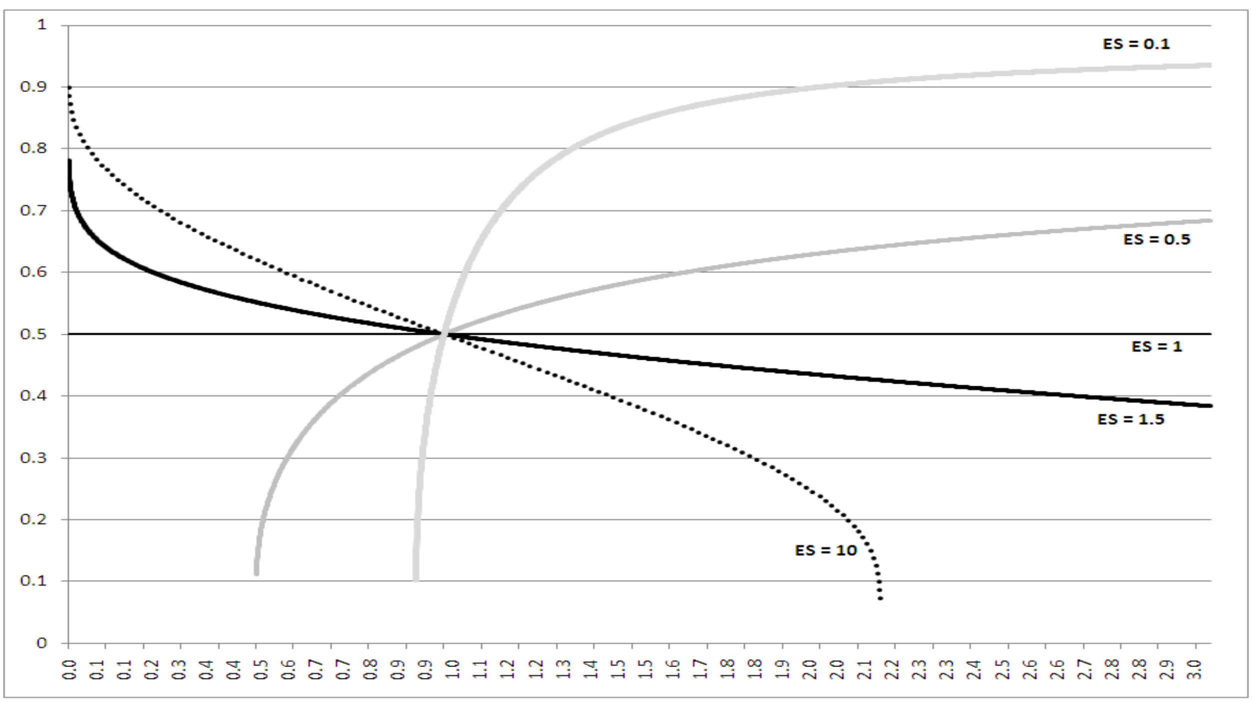
For ES levels close to one (between 0.5 and 1.5), the specification (29) in level gives a pretty good approximation of the exact CES function (24) with no visual difference (see Figure 2 and Figure 3) despites an important change in the input ratio that corresponds to a change in the MRS (i.e. in the relative price between inputs) of more than 800%. For lower or higher level of substitution, we can visually see that the isoquant misbehave when far from the reference point. When inputs are close to perfect complement (ES = 0.1), the isoquant increases after a certain point (Figure 1). Nevertheless reaching this point would require a change in the input price ratio of more than 200%. When the ES is higher than one, the isoquant becomes concave at a certain point (Figure 4 and Figure 5). However, here as well the approximation remains quite good around the reference point since the isoquant starts to misbehave only when the change in the input price ratio is higher than 6 700% (resp. 175%) for an ES level of two (resp. ten).

[10] Because the OEs depend on the input ratio, the derivation of the isoquant from a VOE-CD function in level is not as straightforward as the one from a COE-CD function. It requires reformulating Equation (29) as follows: $x_2 = q - \varphi_1 x$ with $X = X_1 / X_2$ and $\varphi_1 = \left(1 + X^{1/\eta - 1} . \tilde{\varphi}_2 / \tilde{\varphi}_1\right)^{-1}$.

Although the VOE-CD function in level is biased, it provides a reasonable approximation of the CES function around the reference point. The reason is that the change in the OE (27) when the input ratio varies is close to the variation in the OE that would allow reproducing exactly the CES function. Dividing the CES function (24) and the VOE-CD function in level (29) by X_2 and equating the two expressions give the specification of the OE that would allow for reproducing exactly the CES function:

$$\varphi_1 = \frac{\ln\left(\tilde{\varphi}_1 + \tilde{\varphi}_2\left(\frac{X_1}{X_2}\right)^{1-1/\eta}\right)}{\ln\left(\frac{X_1}{X_2}\right)^{1-1/\eta}} \tag{30}$$

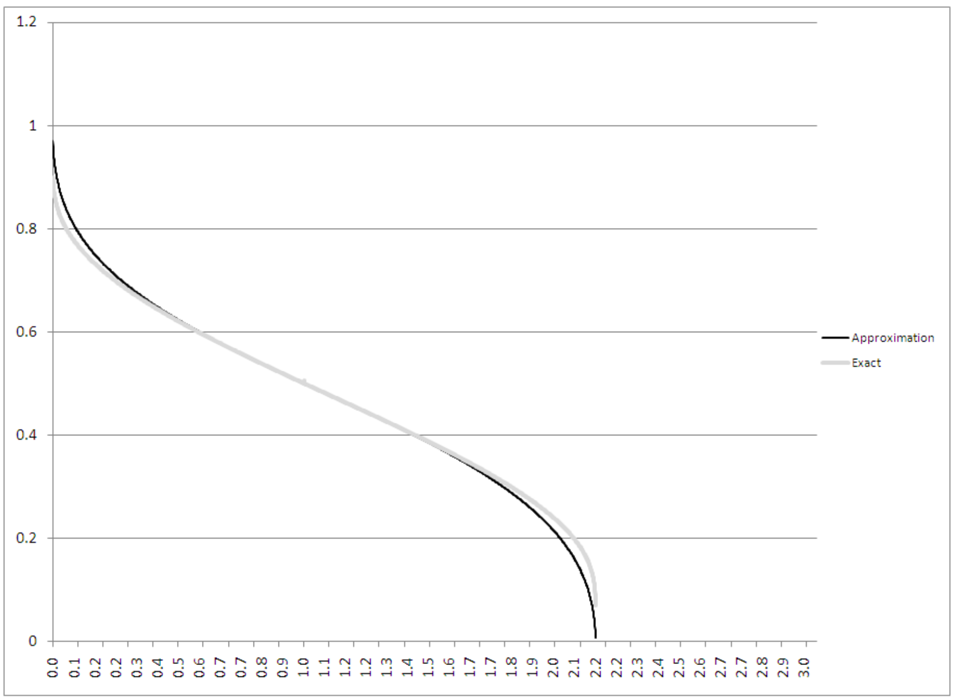
Figure 8. Evolution of the OE for various ES



Key: $Q = 1$; $\tilde{\varphi}_1 = \tilde{\varphi}_2 = 0.5$; X-axis: Input 1 ($X_1$); Y-axis: OE of Input 1 (φ_1) [Equation (30)]; Source: author's calculation.

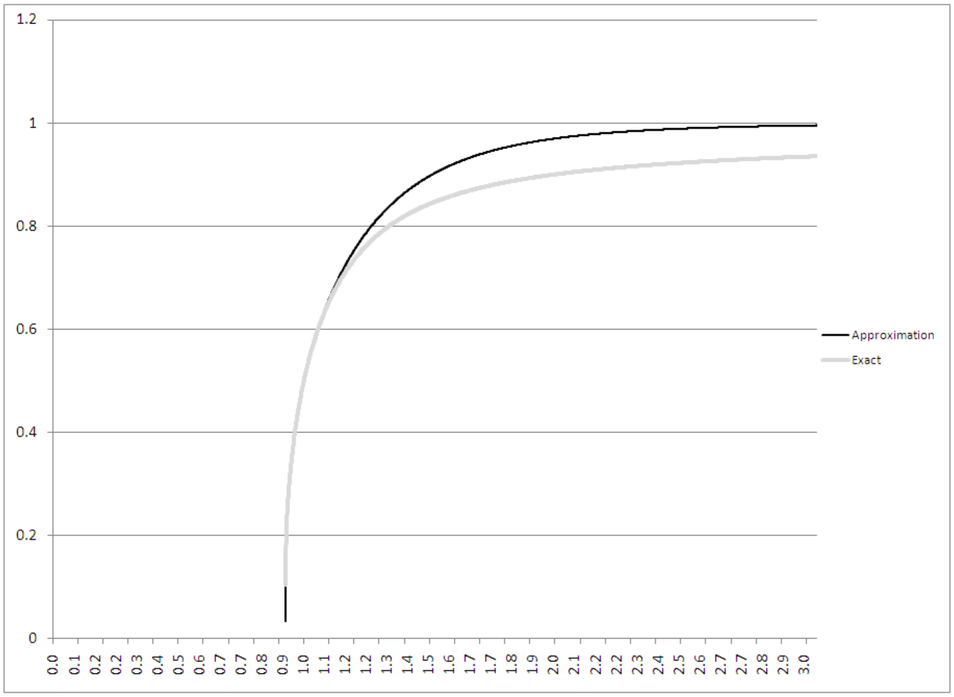
Figure 8 shows the evolutions of the OE (30) for various ES. The X-axis reports the changes in Input 1 which (for a constant level of production) are positively related to the changes in the ratio between Input 1 and Input 2 (X_1 / X_2). When the ratio between Input 1 and Input 2 increases, the OE of Input 1 increases (resp. decreases) if the ES is lower (resp. higher) than one. All curves cross at the reference point where the input ratio is equal to one. The OE (27) follows a very similar pattern (Figure 9 and Figure 10) although noticeable differences appear when the distance from the reference point increases. This error of approximation is the reason for the iso-production curve misbehavior that we have seen above.

Figure 9. Evolution of the OE for ES = 10



Key: $Q = 1$; $\tilde{\varphi}_1 = \tilde{\varphi}_2 = 0.5$; X-axis: Input 1 ($X_1$); Y-axis: OE of Input 1 (φ_1); Exact: OE of Input 1 calculated with Equation (30); Approximation: OE of Input 1 calculated with Equation (27); Source: author's calculation.

Figure 10. Evolution of the OE for ES = 0.1



Key: idem Figure 9.

6. Conclusions

This article has defined the VOE-CD function and shown that this function can be used to approximate any homogeneous production function. This framework appears to have several advantages. First, it is relatively simple compared to most alternative approaches while allowing a wide range of substitution possibilities. It requires only the use of the first order approximation and thus avoid the tedious algebraic of the second order approximation used in flexible function such as the Translog function. It allows for easily respecting the theoretical curvature conditions of the isoquants and for the derivation of linear input demand functions without involving the duality theorem which holds only at the optimum. Second, it provides a generalization of the CES to the case where the ES between each pair of inputs are not equal. Third, it provides a theoretical justification of the empirical success of the CD function usually found in the literature and supports its use for the analysis of long run macroeconomic phenomena.

Moreover, this approach has potentially several very useful applications. As it leads to linear input demands with well-behaved isoquants that are general in terms of substitution possibilities, it may prove promising in the econometric analysis of the producer. In this respect, the recent attempts made by Lemoine et al. (2010) in a related research to estimate a VOE-CD function in the Euro Zone in the case of two inputs (labor and capital) gave promising results.

Applied general equilibrium models provide another important application. As in the multi-sector macroeconomic model Three-ME (Reynès et al., 2011), the input demands derived from a VOE-CD function can easily be introduced in these models. Compared to the nested CES approach, it allows for testing alternative substitution hypothesis without changing the nest of the model. Compared to the use of flexible functions, it has the advantage of tractability since it easily provides well-behaved input demands and does not require the use of temporal data which often are not available.

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